J. Math. Anal. Appl. 462 (2018) 542–553

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Frequently hypercyclic operators with irregularly visiting orbits ☆

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ARTICLE INFO

Article history:
Received 9 November 2017
Available online 9 February 2018
Submitted by J.A. Ball

Keywords:
Hypercyclic and frequently hypercyclic operators and vectors
Irregularly visiting orbits
Frequent Hypercyclicity Criterion
Universal operators

ABSTRACT

We prove that a bounded operator T on a separable Banach space X satisfying a strong form of the Frequent Hypercyclicity Criterion (which implies in particular that the operator is universal in the sense of Glasner and Weiss) admits frequently hypercyclic vectors with *irregularly visiting orbits*, i.e. vectors $x \in X$ such that the set $\mathscr{N}_T(x, U) = \{n \geq 1\,;\, T^n x \in U\}$ of return times of x into U under the action of T has positive lower density for every non-empty open set $U \subseteq X$, but there exists a non-empty open set $U_0 \subseteq X$ such that $\mathscr{N}_T(x, U_0)$ has no density.

1. Introduction

Let X be a separable infinite-dimensional Banach space, and let $\mathscr{B}(X)$ be the space of bounded linear operators on X. We are interested in this paper in the study of dynamics of certain operators $T \in \mathscr{B}(X)$, and the existence of vectors whose iterates under the action of T have an "irregular" behavior. Given $T \in \mathscr{B}(X)$, $x \in X$, and $U \subseteq X$ a non-empty open set, we denote by $\mathscr{N}_T(x, U) = \{n \geq 1\,;\, T^n x \in U\}$ the set of return times of x into U under the action of T. The operator T is said to be *hypercyclic* when there exists a vector $x \in X$ with dense orbit under the action of T, i.e. when $\mathscr{N}_T(x, U)$ is non-empty for every non-empty open set $U \subseteq X$, and *frequently hypercyclic* (resp. *$\mathscr{U}$-frequently hypercyclic*) when there exists $x \in X$ such that $\underline{\text{dens}}\, \mathscr{N}_T(x, U) > 0$ (resp. $\overline{\text{dens}}\, \mathscr{N}_T(x, U) > 0$) for every non-empty open set $U \subseteq X$. We denote respectively by $\underline{\text{dens}}\, A$, $\overline{\text{dens}}\, A$ and $\text{dens}\, A$ the lower density, the upper density, and if it exists, the density of a subset A of $\mathbb{N}$:

$$\underline{\text{dens}}\, A = \liminf_{N \to +\infty} \frac{1}{N} \#\big([1, N] \cap A\big)$$

☆ This work was supported in part by EU IRSES grant AOS (PIRSES-GA-2012-318910), by the Labex CEMPI (ANR-11-LABX-0007-01), and by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).

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https://doi.org/10.1016/j.jmaa.2018.02.020
0022-247X/

$$\overline{\text{dens}}\, A = \limsup_{N\to+\infty} \frac{1}{N}\, \#\big([1,N]\cap A\big)$$

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Vectors x with one of the properties above are called respectively hypercyclic, frequently hypercyclic, and $\mathscr{U}$-frequently hypercyclic vectors. We denote by $\text{HC}(T)$, $\text{FHC}(T)$, and $\text{UFHC}(T)$ these three sets of vectors. When T is hypercyclic, $\text{HC}(T)$ is a dense G_δ subset of X. The set $\text{FHC}(T)$, although dense in X, is meager in X for every frequently hypercyclic operator T ([15], [4], see also [11]), while $\text{UFHC}(T)$ is comeager in X for every $\mathscr{U}$-frequently hypercyclic operator T ([4]). The notion of frequent hypercyclicity was introduced in the paper [2], while $\mathscr{U}$-frequent hypercyclicity was first considered by Shkarin in [17]. These two concepts have been the object of an important amount of work in recent years. We refer the reader to the books [13] and [3], as well as to the papers [11], [5], [6], [14] and [12] (among many others) for an in-depth study of frequent and $\mathscr{U}$-frequent hypercyclicity and related phenomena.

There are (as of now) essentially two known ways of constructing frequently hypercyclic vectors for an operator $T \in \mathscr{B}(X)$: by an explicit construction, or by using ergodic theory. In the first approach, explicit frequently hypercyclic vectors are constructed as a series of vectors whose iterates have suitable properties. The most classical construction of his type is the one yielding the so-called Frequent Hypercyclicity Criterion, first proved in [2] and then generalized in [7]. Another explicit construction of such vectors, making use of some assumptions concerning the periodic points of the operator, is given in [12]. In particular, operators with the so-called Operator Specification Property, which were shown in [1] to be frequently hypercyclic, satisfy this criterion. The second, widely used, approach for proving that a given operator is frequently hypercyclic, is to use ergodic theory: one shows that the operator is *ergodic* in the sense that it admits an invariant probability measure m with full (topological) support with respect to which it defines an ergodic transformation of the space. An application of Birkhoff's pointwise ergodic theorem then shows that T is frequently hypercyclic. More precisely, if $T \in \mathscr{B}(X)$ is ergodic with respect to a probability measure m on X such that $m(U) > 0$ for every non-empty open set $U \subseteq X$, then, for every such $U \subseteq X$, $\text{dens}\, \mathscr{N}_T(x,U) = m(U)$ for m-almost every $x \in X$. It follows by considering a countable basis $(U_p)_{p\ge1}$ of non-empty open subsets of X that m-almost every vector $x \in X$ is frequently hypercyclic for T, and satisfies $\text{dens}\, \mathscr{N}_T(x,U_p) = m(U_p)$ for every $p \ge 1$. Hence the following question, which was posed in a first version, dating from 2013, of the survey paper [10], arises naturally:

Question 1.1. Does there exist a frequently hypercyclic operator T on a Banach space X which admits a frequently hypercyclic vector $x \in X$ such that for some non-empty open set $U \subseteq X$, the set $\mathscr{N}_T(x,U)$ has no density, i.e. $\underline{\text{dens}}\, \mathscr{N}_T(x,U) < \overline{\text{dens}}\, \mathscr{N}_T(x,U)$?

A related question is due to Shkarin, who asked in [17] whether all frequently hypercyclic operators $T \in \mathscr{B}(X)$ admit a frequently hypercyclic vector x such that, for every non-empty open set $U \subseteq X$, $\mathscr{N}_T(x,U)$ contains a set of positive density. As mentioned above, all ergodic operators satisfy this property.

Observe that Question 1.1 is very easy to answer if one withdraws the requirement that the vector $x \in X$ be frequently hypercyclic for T. Indeed, if $T \in \mathscr{B}(X)$ is any $\mathscr{U}$-frequently hypercyclic operator, $\text{UFHC}(T)$ is comeager in X while $\text{FHC}(T)$ is meager. Hence $\text{UFHC}(T) \setminus \text{FHC}(T)$ is comeager in X, and any vector x belonging to this set has the property that for some non-empty open set $U \subseteq X$, $\underline{\text{dens}}\, \mathscr{N}_T(x,U) = 0$ while $\overline{\text{dens}}\, \mathscr{N}_T(x,U) > 0$.

A first progress concerning Question 1.1 was made by Y. Puig de Dios in [16]. He proved there the following result: for any frequently hypercyclic operator T on a Banach space X, any frequently hypercyclic vector $x \in X$ and any non-empty open subset U of X with the property that none of the sets $\bigcup_{n=0}^{N} T^{-n}U$, $N \ge 0$, is dense in X, the set $\mathscr{N}_T(x,U)$ has different lower density and upper Banach density.