J. Math. Anal. Appl. 461 (2018) 1067–1083

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Uniqueness of conservative solutions to the modified two-component Camassa–Holm system via characteristics

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ARTICLE INFO

Article history:
Received 12 October 2017
Available online 2 February 2018
Submitted by D. Wang

Keywords:
Modified two-component Camassa–Holm system
Uniqueness
Characteristic

ABSTRACT

In this paper, we prove the uniqueness of conservative solutions to the modified two-component Camassa–Holm system by the analysis of characteristics.

1. Introduction

In this paper, we consider the Cauchy problem of the following modified two-component Camassa–Holm system

$$\begin{cases} m_t + um_x + 2mu_x = -\eta\bar{\eta}_x, \\ \eta_t + (\eta u)_x = 0, \\ (m,\eta)(0,x) = (m_0,\eta_0)(x), \end{cases} \tag{1.1}$$

where $m = u - u_{xx}$, $\eta = (1-\partial_x^2)(\bar{\eta}-\bar{\eta}_0)$. The system (1.1) proposed by Holm et al. in [21] is a modified version of the two-component Camassa–Holm (CH2) shallow water system to allow a dependence on not only the average density $\bar{\eta}$ but also the pointwise density η. (1.1) is written in terms of velocity u and locally averaged density $\bar{\eta}$. Here $\bar{\eta}_0$ is taken to be a constant.

The mathematical properties of the two types of two-component Camassa–Holm system have been investigated in many works. The CH2 system was derived as a model for shallow water by Constantin and Ivanov

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https://doi.org/10.1016/j.jmaa.2018.01.057
0022-247X/

in [5]. This generalization possessed the peakon, multi-kink solutions and the bi-Hamiltonian structure [3,8]. Well-posedness and wave breaking mechanism were discussed in [1,2,4–6,9–11,13,16–18,22].

The modified two-component Camassa–Holm (MCH2) system can be defined as geodesic motion on the semidirect product Lie group with respect to a certain metric and be given as a set of Euler–Poincaré equations on the dual of the corresponding Lie algebra, cf. [20,21]. Recently, the MCH2 system have been studied in [6,12,14,15,19,23] and the references therein. In [6] the authors proved that the system is locally well-posed and presented some blow-up results. By using Helly theorem and some a priori one-sided upper bound and higher integrability space–time estimates on the first-order derivatives of approximation solutions, Guan and Yin [14] obtained the existence of global-in-time weak solutions. Guo and Zhu [19] established sufficient conditions on the initial data to guarantee blow-up solutions. In [23], Tan and Yin proved the existence of global conservative solutions to the Cauchy problem (1.1). However, the uniqueness of this global weak solutions have not been discussed yet.

The main purpose of this paper is to prove the uniqueness of the conservative solutions obtained in [23]. Given a conservative solution, we introduce a set of auxiliary variables tailored to this solution, and show that these variables satisfy a suitable semilinear system of equations. By proving that this semilinear system has a unique solution, we eventually obtain the uniqueness of the conservative solution to the original equation (2.1).

The rest of this paper is organized as follows. In Section 2, we reformulate the problem (1.1), review basic definitions and state our main uniqueness result. In Section 3, we prove the uniqueness of characteristic. The proof of our main theorem will be given in the last section.

At the end of this section, we introduce some notations.

Notations: Throughout this paper, for simplicity, we will omit the variables t, x of functions if it does not cause any confusion. C denotes a generic positive constant which may vary in different estimates. We denote $*$ the spatial convolution. Since all space of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity.

2. Basic definitions and results

In this section, we first reformulate the problem (1.1), then we recall some useful estimates and the existence result of global conservative solution to (1.1), cf. [23]. Our main uniqueness result is stated at the end of this section. To begin with, introducing a new variable $\rho = \bar{\eta} - \bar{\eta}_0$, so that $\eta = \rho - \rho_{xx}$. Then the system (1.1) is equivalent to the following

$$\begin{cases} u_t + uu_x = -\partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - \frac{1}{2}\rho_x^2\right), \\ \rho_t + u\rho_x = -\partial_x p * (u_x \rho_x) - p * (u_x \rho), \\ (u, \rho)(0, x) = (u_0, \rho_0)(x), \end{cases} \tag{2.1}$$

with initial data $(u_0, \rho_0)(x) \in H^1 \times (H^1 \cap W^{1,\infty})$. Here we have used the fact that if $p(x) := \frac{1}{2}e^{-|x|}, x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$. For convenience, we set

$$\begin{aligned} P^1 &= p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - \frac{1}{2}\rho_x^2\right), \\ P^2 &= p * (u_x \rho_x), \\ P^3 &= p * (u_x \rho). \end{aligned} \tag{2.2}$$

Following [23], we introduce the definition of a conservative solution of (2.1).