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Volterra type operators on $S^p(\mathbb{D})$ spaces ☆

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ABSTRACT

Čučković and Paudyal characterized the lattice of invariant subspaces of the operator T in the Hardy–Hilbert space $H^2(\mathbb{D})$, where they studied the special case (when $p = 2$) of the space $S^p(\mathbb{D})$. We generalize some of their works to the general case when $1 \leq p < \infty$ and determine that $\mathcal{M}$ is an invariant subspace of T on $H^p(\mathbb{D})$ if and only if $T_z(\mathcal{M})$ is an invariant subspace of M_z on $S_0^p(\mathbb{D})$, if and only if $T_z(\mathcal{M})$ is a closed ideal of $S_0^p(\mathbb{D})$. Furthermore, we provide certain Beurling-type invariant subspaces of M_z on $S^p(\mathbb{D})$ and $S_0^p(\mathbb{D})$. Then, we investigate the boundedness of the operators T_g and I_g on $S^p(\mathbb{D})$. Finally, we investigate the spectrum of multiplication operator M_g on $S^p(\mathbb{D})$, the isometric multiplication operators and the isometric zero-divisors on $S^p(\mathbb{D})$.

1. Introduction

The study of the invariant subspace problem, namely the question of the existence of nontrivial invariant subspaces for bounded linear operators, extends back to the work of von Neumann. While Enflo [13] and Read [20,21] have demonstrated that several operators exist on Banach spaces without nontrivial invariant subspaces, it remains unknown whether every bounded linear operator on an infinite-dimensional Hilbert space has a nontrivial invariant subspace. An important breakthrough was made by Beurling's paper [7] in 1949, in which he completely characterized the invariant subspaces of the shift operator on H^2.

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space consisting of all analytic functions on the unit disk. For $0 < p < \infty$, the analytic Hardy space $H^p(\mathbb{D})$ on the unit disk $\mathbb{D}$ consists of all analytic functions $f \in H(\mathbb{D})$ satisfying

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$$\|f\|_{H^p(\mathbb{D})} = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty\,.$$

Moreover, for $p = \infty$, the space H^∞ is defined by

$$H^\infty = \{f \in H(\mathbb{D}) : \ \|f\|_\infty \equiv \|f\|_{H^\infty} \equiv \sup_{z\in\mathbb{D}}\{|f(z)|\} < \infty\}\,.$$

Here, we define several operators on $H^p(\mathbb{D})$. The shift operator on $H^p(\mathbb{D})$ is defined by

$$(M_z f)(z) = zf(z) \text{ for any } f \in H^p(\mathbb{D}) \text{ and } z \in \mathbb{D}\,.$$

For any analytic function $g \in H(\mathbb{D})$, the Volterra type operator T_g is defined by

$$(T_g f)(z) = \int_0^z f(\omega)g'(\omega)d\omega\,,$$

for any $f \in H^p(\mathbb{D})$. When $g(z) = z$, $T_z = \int_0^z f(\omega)d\omega$ is the simplest Volterra operator. An integral operator related to T_g (denoted by I_g) is defined by

$$(I_g f)(z) = \int_0^z f'(\omega)g(\omega)d\omega\,,$$

for any $f \in H^p(\mathbb{D})$.

For any $g \in H(\mathbb{D})$, the multiplication operator M_g on $H^p(\mathbb{D})$ is defined by

$$(M_g f)(z) = f(z)g(z)\,,$$

for any $f \in H^p(\mathbb{D})$. Then, M_g is related to T_g and I_g by

$$(M_g f)(z) = f(0)g(0) + (T_g)f(z) + (I_g)f(z)\,.$$

We introduce an additional operator, as follows:

$$(Tf)(z) = (M_z f)(z) + (T_z f)(z)\,, \text{ for } f \in H^p(\mathbb{D}) \text{ and } z \in \mathbb{D}\,.$$

Sarason [23] studied the lattice of invariant subspaces of the operator T acting on $L^2(0,1)$. Following his work, Čučković and Paudyal [11] characterized the lattice of invariant subspaces of the operator T on the Hardy–Hilbert space $H^2(\mathbb{D})$, and studied the special case (when $p = 2$) of the space $S^p(\mathbb{D})$, defined by:

$$S^p(\mathbb{D}) = \{f \in H^p(\mathbb{D}) : Df \in H^p(\mathbb{D})\},$$

where $D = \frac{d}{dz}$ is the differential operator. Two norms exist for $S^p(\mathbb{D})$, which are respectively defined by

$$\|f\|_{S^p(\mathbb{D})} = \|f\|_{H^p(\mathbb{D})} + \|Df\|_{H^p(\mathbb{D})},$$

and

$$\|f\|_{S^p(\mathbb{D})} = |f(0)| + \|Df\|_{H^p(\mathbb{D})}\,,$$

and known as the first and second norms for $S^p(\mathbb{D})$, respectively.