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On commuting solutions of the Yang–Baxter-like matrix equation ☆

Dongmei Shen [a,*], Musheng Wei [b], Zhigang Jia [c]

[a] School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China
[b] College of Mathematics and Science, Shanghai Normal University, Shanghai 200234, China
[c] School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China

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ABSTRACT

Let A be an arbitrary square matrix and its Jordan canonical form is $P^{-1}AP = J = \mathrm{diag}(J_1(\lambda_1), \cdots, J_q(\lambda_q))$ with P an invertible matrix. $\lambda_1, \cdots, \lambda_q$ are different eigenvalues of matrix A. The commuting solution problem of the matrix equation $AXA = XAX$ is equivalent to the problem $J_i(\lambda_i)Y^{(i)} = Y^{(i)}J_i(\lambda_i)$, $J_i(\lambda_i)^2Y^{(i)} = J_i(\lambda_i)(Y^{(i)})^2$ with $Y = P^{-1}XP = \mathrm{diag}(Y^{(1)}, \cdots, Y^{(q)})$. We give the structures of the commuting solutions $Y^{(i)}$ in special Toeplitz forms. Based on them, we construct new matrices $H_\eta^{(\delta_1,\delta_2)}$ related to the commuting solutions. Then we propose a method of solving all the commuting Yang–Baxter-like solutions, by which all solutions can be obtained step by step by recursively solving matrix equations in two cases $\lambda_i = 0$ or $\lambda_i \neq 0$ with respect to the i-th Jordan block $J_i(\lambda_i)$.

1. Introduction

In 1967, C.N. Yang, a Nobel laureate, suggested a matrix equation in his paper concerning the many-body problem in one dimension with repulsive delta-function interaction [21],[22]. In 1972, R.J. Baxter, a statistical physicist, also obtained the same equation in his work concerning statistical model with six-vertices [4]. By solving the matrix equation in terms of some simple solutions, both of them could obtain exact, analytical results of the physical systems considered by them. Later on, it was realized that the equation also appears in many branches of physics and mathematics, like statistical physics, field theory, and quantum integrable systems in low dimensions, quantum groups, topology in three dimension, knot theory and so on. Over the past two decades, it is playing an ever increasingly important role in the fields alluded to above [1–9],

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* Corresponding author.
E-mail addresses: shendongmei@sfu.edu.cn (D. Shen), mwei@shnu.edu.cn (M. Wei), zhgjia@jsnu.edu.cn (Z. Jia).

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[23]. The equation attracted the attention from many physicists and mathematicians and is now an actively pursued research field. The equation is called the Yang–Baxter equation.

In the context of a unital associative algebra U with unit e, the original quantum parameter-free Yang–Baxter equation is an equation for an invertible element R of the tensor product $U \otimes U$. It has the form

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1}$$

where $R_{12} = \phi_{12}(R), R_{13} = \phi_{13}(R), R_{23} = \phi_{23}(R)$. Here $\phi_{12}, \phi_{13}, \phi_{23}$ are algebra morphisms from $U \otimes U$ to $U \otimes U \otimes U$, defined by

$$\phi_{12}(u, v) = u \otimes v \otimes e, \quad \phi_{13} = u \otimes e \otimes v, \quad \phi_{23} = e \otimes u \otimes v.$$

In its simpler parameter-free format, the original Yang–Baxter equation can be written in the form of the quadratic matrix equation

$$AXA = XAX, \tag{2}$$

which may be called the Yang–Baxter-like matrix equation. In (2), the given A and the unknown X are $n \times n$ complex matrices.

Although some solutions have been found for Yang–Baxter equation in quantum group theory, no systematical study of (2) has appeared in the literature as a purely linear algebra problem. One possible reason is that solving a polynomial system of n^2 quadratic equations with n^2 unknowns is a challenging topic. Some special cases of (2) have been discussed in [10–12], [18], [20], [24]. The first attempt of solving (2) from the angle of matrix theory is the paper [10], in which Brouwer's fixed point theorem was applied to find a nontrivial solution when A is an invertible row-sum one matrix such that A^{-1} is a stochastic one. Then in [11], spectral solutions of the matrix equation (2) were given in terms of the spectral projectors. In [12], the authors continued to study the spectral solutions of (2) by constructing infinitely many projection-based solutions with respect to one eigenvalue. In contrast to the works in [11], the authors focused on the possibility of finding infinitely many solutions for the special case that the corresponding eigenvalue is semisimple with multiplicity at least two. In [18], the authors have found the expression of all solutions of (2) when A is a given idempotent matrix. In [20] and [24], all solutions of (2) have been obtained when A has rank one or rank two, respectively.

On the other hand, in [13],[14] and [15], commuting solutions of (2) have been proposed, which means that the unknown matrix X of (2) can commute with A, namely, $AX = XA$. In [13], the authors have found all the commuting solutions of (2) when A has some special Jordan structure. In which, they have discussed three cases for each Jordan block, such as the i-th $n_i \times n_i$ Jordan block $J^{(i)} = \lambda_i I$, $J^{(i)} = J_{n_i}(\lambda_i)$ and $J^{(i)} = \mathrm{diag}(\lambda_i, J_{n_i-1}(\lambda_i))$. In [14], the authors have found all commuting solutions of (2) when A is diagonalizable. In [15], the authors have obtained the structure of all the commuting solutions of (2) when A is an arbitrary nilpotent matrix. If A is a matrix with any Jordan structure forms, how to find the commuting solutions of (2) is also an unsolved problem.

In this paper, we focus on the possibility of finding all commuting solutions of (2) when A is any square matrix with general Jordan structure forms. Suppose the Jordan canonical form of A is $P^{-1}AP = J = \mathrm{diag}(J_1(\lambda_1), \cdots, J_q(\lambda_q))$ with P an invertible matrix. $\lambda_1, \cdots, \lambda_q$ are different eigenvalues of matrix A. The commuting solution problem of the Yang–Baxter-like matrix equation $AXA = XAX$ is equivalent to the problem $J_i(\lambda_i)Y^{(i)} = Y^{(i)}J_i(\lambda_i), J_i(\lambda_i)^2Y^{(i)} = J_i(\lambda_i)(Y^{(i)})^2$ with $Y = P^{-1}XP = \mathrm{diag}(Y^{(1)}, \cdots, Y^{(q)})$. We obtain the structures of the commuting solutions $Y^{(i)}$ in special Toeplitz forms. Based on above observations, we construct new matrices $H_\eta^{(\delta_1,\delta_2)}$ related to the commuting solutions. Then we propose a method of solving all commuting solutions, by which we can obtain all commuting solutions step by step by recursively solving matrix equations in two cases $\lambda_i = 0$ or $\lambda_i \neq 0$ with respect to the i-th Jordan block $J_i(\lambda_i)$.