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Global existence of solutions for semi-linear wave equation with scale-invariant damping and mass in exponentially weighted spaces

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A B S T R A C T

In this paper we consider the following Cauchy problem for the semi-linear wave equation with scale-invariant dissipation and mass and power non-linearity:

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{\mu_2^2}{(1+t)^2} u = |u|^p, \quad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \ (\star)$$

where μ_1, μ_2^2 are nonnegative constants and $p > 1$. On the one hand we will prove a global (in time) existence result for $(\star)$ under suitable assumptions on the coefficients μ_1, μ_2^2 of the damping and the mass term and on the exponent p, assuming the smallness of data in exponentially weighted energy spaces. On the other hand a blow-up result for $(\star)$ is proved for values of p below a certain threshold, provided that the data satisfy some integral sign conditions. Combining these results we find the critical exponent for $(\star)$ in all space dimensions under certain assumptions on μ_1 and μ_2^2. Moreover, since the global existence result is based on a contradiction argument, it will be shown firstly a local (in time) existence result.

1. Introduction

In this paper we study the global in time existence of small data solutions and the blow-up in finite times of solutions to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{\mu_2^2}{(1+t)^2} u = |u|^p, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

in any space dimension $n \geq 1$, where $\mu_1 > 0$ and $\mu_2^2 \geq 0$ are constants and $p > 1$.

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The main purpose of the present article is to extend some results of [18] to any spatial dimension. More precisely, considering the quantity

$$\delta := (\mu_1 - 1)^2 - 4\mu_2^2 \tag{1.2}$$

which describes somehow the interplay between the damping and the mass term, in [18] it is proved a global existence result of small data energy solutions in space dimensions $n = 1, 2, 3, 4$ for a certain range of δ, assuming additional L^1 regularity for the initial data. The restriction on the spatial dimension is due to the employment of Gagliardo–Nirenberg inequality in the estimates of the non-linear term, which implies the restrictions $p \geq 2$ and $p \leq p_{\mathrm{GN}}(n) := \frac{n}{n-2}$ when $n \geq 3$ for the exponent of the non-linearity. In order to avoid this type of conditions on p, we may consider stronger assumptions on initial data. More specifically, we will consider data in energy spaces with weight of exponential type.

Let us clarify the role of δ in the description of the equation

$$u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{\mu_2^2}{(1+t)^2} u = 0. \tag{1.3}$$

Considering $v(t,x) = (1+t)^\gamma u(t,x)$, we can transform (1.3) in a wave equation with either just a scale-invariant damping or just a scale-invariant mass with a suitable choice of $\gamma \in \mathbb{R}$, depending on the value of δ.

In particular for $\delta \geq (n+1)^2$ we can transform (1.3) in a scale-invariant wave equation with a damping term, which looks like effective under the point of view of decay estimates (even though, according to the classification introduced in [24], the scale-invariant time dependent coefficient of the damping term for the transformed equation does not belong to the class of effective damping terms). Roughly speaking, assuming the above mentioned range for δ, we guarantee $L^2 - L^2$ estimates for (1.3) of “parabolic type”for the solution and its first order derivatives.

Let us report a brief historical overview, that is functional to elucidate our approach, on those papers in which this type of exponentially weighted Sobolev spaces is used in the study of semi-linear hyperbolic equations.

A first pioneering work in this direction is represented by [22], in which a global existence (in time) result in the space $\mathcal{C}([0,\infty), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^n))$ is proved for a classical damped wave equation with power non-linearity $|u|^p$, provided that the exponent satisfies $p > p_{\mathrm{Fuj}}(n) := 1 + \frac{2}{n}$ and $p \leq p_{\mathrm{GN}}(n)$ for $n \geq 3$ and the data are compactly supported in $B_K(0) = \{x \in \mathbb{R}^n : |x| \leq K\}$. In particular, the weight used in the derivation of this result is $\mathrm{e}^{\psi_0(t,\cdot)}$, where

$$\psi_0(t,x) = \tfrac{1}{2}\big(t + K - \sqrt{(t+K)^2 - |x|^2}\big) \quad \text{for } |x| < t + K.$$

Afterwards, in [12] the authors improved, for the same range of p, the previous result for the classical damped wave equation, removing the compactness assumption for the support of data and requiring instead the belonging of data to certain exponentially weighted spaces. This goal is achieved through a different choice of the weight function. Namely, instead of ψ_0 the authors consider

$$\psi_1(t,x) = \tfrac{|x|^2}{4(1+t)}.$$

Subsequently, the damped wave equation with coefficient $b(t) = b_0(1+t)^{-\beta}$, where $b_0 > 0$, was studied for absorbing non-linearity (when $-1 < \beta < 1$) and for power non-linearity (whether $0 \leq \beta < 1$) in [17] and [16], respectively, for a suitable choice of the exponent function ψ.

Indeed, in [17] some a-priori estimates are derived for solutions to the equation

$$u_{tt} - \Delta u + b(t)u_t = -|u|^{p-1}u$$