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Singular Hamiltonian system with several spectral parameters

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A B S T R A C T

In this paper, the Weyl–Titchmarsh theory has been constructed for the singular 2n-dimensional (even order) Hamiltonian system with several spectral parameters. In particular, we consider that the left end point of the interval is regular and the right end point of the interval is singular for the Hamiltonian system with several parameters. Using the nested circles approach, we prove that at least n-linearly independent solutions are squarly integrable with respect to some matrix functions.

1. Introduction

Differential equations with many spectral parameters can be handled on a single interval, as well as on the product of many intervals with appropriate boundary conditions. Such problems are called multiparameter eigenvalue problems. If the problem is constructed with many equations, then the solution will be the product of each solution of the corresponding differential equation. Regular multiparameter eigenvalue problems can be found in, for example, [2–7,29–31]. Moreover, some singular problems have been studied in [3,11,14,15,20].

In 1973, Sleeman [28] investigated the following singular second order differential equations

$$-\frac{d^2y_r}{dx_r^2}+\sum_{s=1}^{k}\{p_{rs}(x_r)\lambda_r+q_r(x_r)\}\,y_r(x_r)=0,\ x_r\in[a_r,b_r), \tag{1.1}$$

where $r=1,...,k$, a_r is the regular point and b_r is the singular point for the rth differential equation in (1.1), $p_{rs}(x_r)$, $q_r(x_r)$ are real-valued and continuous functions on $[a_r,b_r)$, $\det\{p_{rs}(x_r)\}_{r,s=1}^{k}>0$ for all $(x_1,...,x_k)\in(a_1,b_1)\times...\times(a_k,b_k)$, and λ_s are the spectral parameters, and generalized the results of Weyl [33] obtained for a single singular second order differential equation with a single spectral parameter given in (1.1). In fact, Sleeman firstly handled the equation

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$$-\frac{d^2y}{dx^2}+\{\lambda_1 p_1(x)+\lambda_2 p_2(x)+q(x)\}\,y=0,\ \ x\in[a,b), \tag{1.2}$$

where a is the regular point and b is the singular point for (1.2). After some modifications he considered the problem (1.2) as

$$-\frac{1}{\rho(x)}\frac{d^2y}{dx^2}+\{\lambda_1 s_1(x)+\lambda_2 s_2(x)+t(x)\}\,y=0,\ \ x\in[a,b), \tag{1.3}$$

where

$$\rho(x)=\epsilon\,[\alpha_1 p_1(x)+\alpha_2 p_2(x)]>0,\ \ s_k(x)=\frac{\alpha_k p_k(x)}{\rho(x)},\ \ t(x)=\frac{q(x)}{\rho(x)},$$

$k=1,2$, $\epsilon=\pm1$, α_k are some real numbers. Reformulation (1.3) introduce some symmetry and provides some flexibility in the analysis. Then he proved that the solution

$$\psi(x;\lambda_1,\lambda_2)=\phi(x;\lambda_1,\lambda_2)M(b;\lambda_1,\lambda_2)+\theta(x;\lambda_1,\lambda_2)$$

of the equation (1.3) satisfies the inequality

$$\left|\operatorname{Im}\int_a^b |\psi|^2\left(\lambda_1 s_1+\lambda_2 s_2\right)\rho(x)dx\right|<\infty,$$

where ϕ and θ satisfy the following initial conditions

$$\begin{array}{ll}\phi(a;\lambda_1,\lambda_2)=\sin\alpha, & \phi'(a;\lambda_1,\lambda_2)=-\cos\alpha,\\ \theta(a;\lambda_1,\lambda_2)=\cos\alpha, & \theta'(a;\lambda_1,\lambda_2)=\sin\alpha,\end{array}$$

$\alpha\in[0,\pi)$ and $\operatorname{Im}(\lambda_1 s_1+\lambda_2 s_2)$ is of one sign and non-zero for all $x\in[a,b)$. Here $M(b;\lambda_1,\lambda_2)$ is a point which is inside all of the circles $C_b(\lambda_1,\lambda_2)$ or on a circle $C_b(\lambda_1,\lambda_2)$. The radius of $C_b(\lambda_1,\lambda_2)$ is described by

$$r_b(\lambda_1,\lambda_2)=-\frac{1}{2}\left\{\int_a^b \rho\left(\operatorname{Im}\lambda_1 s_1+\operatorname{Im}\lambda_2 s_2\right)|\phi|^2\,dx\right\}^{-1}$$

and may be zero in the limit-point case. Moreover, Sleeman gave some examples to show that although

$$\int_a^b \rho\left(\operatorname{Im}\lambda_1 s_1+\operatorname{Im}\lambda_2 s_2\right)|\phi|^2\,dx=\infty,$$

it may happen that

$$\int_a^b \rho\,|\phi|^2\,dx<\infty,$$

and this is still a limit-point situation. Note that Sims in his work on nonselfadjoint one-parameter problems gave two examples which substantiate this situation [27]. Consequently, Sleeman obtained the following cases under the consideration that $\operatorname{Im}(\lambda_1 s_1+\lambda_2 s_2)$ is of one sign and non-zero for all $x\in[a,b)$: