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Nonisotropic chaotic oscillations of the wave equation due to the interaction of mixing transport term and superlinear boundary condition

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A R T I C L E I N F O

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A B S T R A C T

This paper studies the nonisotropic chaotic oscillations of the initial-boundary value problem of one-dimensional wave equation with a mixing transport term. It separately considers that the boundary condition at the right-end of the wave equation is a superlinear type and linear perturbation of such type, each causing the total energy of the underlying system to rise and fall due to the interaction with a mixing transport term. For each type of boundary condition, the occurrence of nonisotropic chaotic oscillations is rigorously proved. Numerical examples verify the effectiveness of theoretical prediction.

1. Introduction

Chaotic behaviors in nonlinear dynamical systems of finite-dimensional ordinary differential equations (ODEs) have been extensively studied and the existence of these behaviors has been also rigorously proved (see [18–20,22] and references therein). For systems governed by partial differential equations (PDEs), however, the rigorous proof of the existence of their chaotic behaviors is challenging and theoretical results are very few. Li [14] put forward a standard framework for proving the existence of chaos in a near-integrable PDE. Bernardes et al. [1] characterized distributional chaos in the systems of linear operators on the Fréchet space. Very recently, Yin and Yang [23,24,26,25] further investigated distributional single-chaos and distributional n-chaos in some systems of linear operators including those of the unforced quantum harmonic oscillator and weighted shift operator. For more theoretical results on the existence of chaos in infinite-dimensional dynamical systems including those of PDEs, we refer to [9,10,15,21] and references therein.

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In order to introduce the mathematical models to be studied, let us revisit some related studies and recall some facts. In 1998, Chen et al. [3–5] proved that chaotic oscillations can occur in the system of one-dimensional linear wave equation

$$w_{tt}(x,t) - w_{xx}(x,t) = 0, \quad x \in (0,1), \quad t > 0 \tag{1}$$

with a certain van der Pol boundary condition. It is worth mentioning that such chaotic oscillations are isotropic with respect to space and time since two associated families of characteristics propagate with the same speed (specifically, the "strength of chaos" is the same along both x and t directions). If these two propagation speeds are different, then the corresponding chaotic oscillations are nonisotropic with respect to space and time. Interestingly, Chen et al. [6] showed nonisotropic spatiotemporal chaotic oscillations of the wave equation due to mixing transport and van der Pol boundary condition. Huang [13] also studied nonisotropic chaotic oscillations of the wave equation with a van der Pol boundary condition by means of growth rates of total variations. Li et al. [16] further showed that nonisotropic spatiotemporal chaotic oscillations can exist in the wave equation with a general nonlinear boundary condition restricted to certain assumptions.

One simple kind of nonconservative system with a single degree of freedom is the following ODE

$$\ddot{x} + \varphi(x, \dot{x}) + f(x) = 0, \tag{2}$$

where x is the displacement, $f(x)$ is the potential force, and $\varphi(x, \dot{x})$ is the damping force of the system (2). The total energy E of system (2) is defined by

$$E = \frac{1}{2}\dot{x}^2 + \int f(x)dx.$$

The time change rate of E satisfies

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\dot{x}\varphi(x, \dot{x}),$$

from which we know that the total energy is monotonically decreasing if $\dot{x}\varphi(x, \dot{x}) > 0$ but monotonically increasing if $\dot{x}\varphi(x, \dot{x}) < 0$. However, a number of systems may have the following two characteristics: there is energy loss in some regions of the phase plane, but there is energy supply in the other regions. The damping force $\varphi(x, \dot{x})$ may be a function only with respect to velocity $\dot{x}$ (called the damping force of velocity type, denoted by $\varphi(\dot{x})$). For convenience, we still call it damping force in the following.

Note that the damping force with a symmetric function of velocity can be approximated by the first two terms of the power series. Therefore, for clarity, we only choose one of the expressions:

$$\varphi(\dot{x}) = -\alpha\dot{x} + \beta\dot{x}^3, \quad \alpha, \beta > 0. \tag{3}$$

Note that Eq. (2) with potential force $f(x) = \omega_0^2 x$ ($\omega_0 \neq 0$) and damping force (3) is nothing but the famous van der Pol equation. The change rate of total energy E in such a type of system is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}^2(\alpha - \beta\dot{x}^2),$$

which implies that the total energy will eventually rise and fall between a certain interval of $\dot{x}$. This mechanism of self-regulation characterized by (3), is highly useful in the design and application of servomechanisms in modern automatic control. Moreover, it is the essential cause for the occurrence of chaotic oscillations in a one-dimensional linear wave equation with boundary condition of van der Pol type.