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On normal forms of complex points of codimension 2 submanifolds [☆]

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ABSTRACT

In this paper we present some linear algebra behind quadratic parts of quadratically flat complex points of codimension two real submanifold in a complex manifold. Assuming some extra nondegenericity and using the result of Hong, complete normal form descriptions can be given, and in low dimensions, we obtain a complete classification without any extra assumptions.

1. Introduction

Let $f\colon M^{2n} \hookrightarrow X^{n+1}$ be a real $2n$-dimensional smooth manifold, embedded into an $(n+1)$-dimensional complex manifold (X, J). We assume that f is at least $\mathcal{C}^2$-smooth. A point $p \in M$ is called *CR regular*, if the dimension of the complex tangent space $T_p^C M = df(T_pM) \cap Jdf(T_pM) \subset T_{f(p)}X$ equals the algebraically expected dimension $n-1$. If the complex dimension of $T_p^C M$ equals n, we call such a point *complex point*. Complex points can be seen as the intersection of $Gf(M)$ with the subbundle $\mathbb{C}\mathrm{P}(T^*X, J)$, where $Gf\colon M \to f^*Gr_{2n}TX$ is the Gauss map $p \mapsto [df(T_pM)] \in Gr_{2n}T_{f(p)}X$ (see [23]). By Thom transversality, for generic embeddings, the intersection is transverse and so complex points are isolated by a simple dimension count. We call such transverse complex points *nondegenerate* and we can assign a sign to such an intersection

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(note that the sign can be assigned even in the case of nonorientable M). If the sign is positive, the complex point is called *elliptic*, if it is negative, *hyperbolic*.

Topological structure of complex points was first studied by Lai [23] and, specifically for surfaces, mostly by Forstnerič [10]. Classification of complex points up to isotopy is treated in [25–27]. Most research into complex points has been focused into trying to understand local hulls of holomorphy. This direction was started by Bishop [5] and is now well understood in the case of surfaces through the works of Kenig and Webster [22], Moser and Webster [24] and Huang [18] among others. Global theory (filling spheres with holomorphic discs) was studied by Bedford and Gaveau [1] and Bedford and Klingenberg [2], and has later resulted in many important theorems in symplectic and contact geometry. In higher dimensions, assuming real analyticity, a similar problem of finding an appropriate Levi flat hypersurface that is bounded by the submanifold near the complex point, is treated first in the papers of Dolbeault, Tomasini and Zaitsev [7,8] and to a greater generality by Huang and Yin [19,20] and Fang and Huang [9]. The problem is equivalent to understanding when the manifold can be holomorphically flattened near the complex point.

A closely related topic is trying to understand normal forms for manifolds near real analytic complex points. Again, the situation is well understood in the surface case by the work of Moser and Webster [24] and Gong [11,12], but it seems to be much more intractable in higher dimensions. It is already an interesting problem to completely classify complex points up to their quadratic part.

If $p \in M$ is a complex point for a $\mathcal{C}^2$-embedding $f\colon M^{2n} \to X^{n+1}$, then in some local coordinates near $f(p)$, M is given by an equation of the form

$$w = \overline{z}^T A z + \mathrm{Re}(z^T B z) + o(|z|^2),$$

where A and B are some $n \times n$ complex matrices with B symmetric. By applying a biholomorphic change of coordinates near $f(p)$, while preserving a general structure of the above equation, the pair of matrices (A, B) transforms into $(cP^*AP, \overline{c}P^TBP)$, where $c \in \mathbb{C}$, $|c| = 1$ and P is a nonsingular $n \times n$ matrix. This is more carefully explained in the next section. Classifying complex points up to their quadratic term means finding nice normal form representatives for matrices A and B for this congruence relation, and it reduces to a linear algebra problem. The classification is trivial in the case of $n = 1$, and for $n = 2$ was done by Coffman [6]. Note that if A is positive definite, we can find a nonsingular P, so that $P^*AP = I$ and then use Autonne–Takagi theorem (any complex symmetric matrix is unitary T-congruent to a real diagonal matrix with non-negative entries) to simplify B. For general Hermitian A (even for semi-definite), we cannot simultaneously diagonalize both A and B, as can quickly be seen by a simple example $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 & b \\ b & 0 \end{smallmatrix}\right]$, where $b \neq 0$.

The purpose of this paper is to give a better understanding of the classification of complex points up to their quadratic term for $n > 2$ in the quadratically flat case (when A is Hermitian after a multiplication by a nonzero complex scalar). Assuming that $\det B \neq 0$ in the pair (A, B), the work of Hong [13,14], Hong and Horn [15] and others, immediately gives complete normal form descriptions in all dimensions. We summarize those results in Section 3 (FORM 1 and FORM 2). In both these forms the matrix B is first made into the identity matrix using Autonne–Takagi theorem, and then A is simplified by $*$-congruence using complex orthogonal matrices. Since we often want the matrix A to be in its simplest form, we also introduce FORM 3, where first A is diagonalized using Sylvester's theorem and then B is simplified while preserving A. At the end of this section we also point out a much simpler description in the generic case of quadratically flat complex points (Proposition 3.1). In Section 4, we extend these results in dimensions $n = 3$ and $n = 4$ (see Theorem 4.3), to obtain a complete classification without any extra assumptions on the pair (A, B). While we fail to give a nice form for complete classification in all dimensions, a quite general result is given in Lemma 4.1.