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On the rate of convergence to equilibrium for the linear Boltzmann equation with soft potentials

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A B S T R A C T

In this work we present several quantitative results of convergence to equilibrium for the linear Boltzmann operator with soft potentials under Grad's angular cut-off assumption. This is done by an adaptation of the famous entropy method and its variants, resulting in explicit algebraic, or even stretched exponential, rates of convergence to equilibrium under appropriate assumptions. The novelty in our approach is that it involves functional inequalities relating the entropy to its production rate, which have independent applications to equations with mixed linear and non-linear terms. We also briefly discuss some properties of the equation in the non-cut-off case and conjecture what we believe to be the right rate of convergence in that case.

Contents

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1. Introduction

This work is concerned with the asymptotic behaviour of the linear homogeneous Boltzmann equation in the less explored case of soft potential interactions, and with a cut-off assumption (the precise definition of all the above will be given shortly). We are interested in the application of entropy techniques to study the approach to equilibrium in the relative entropy sense, and in the application of entropy inequalities to estimate its rate. Our results complement a previous work by two of the authors [5], where the case of hard potentials was studied following the same techniques.

Our motivation comes partly from the study of the linear Boltzmann equation itself, which is a basic model in kinetic theory describing the collisional interaction of a set of particles with a thermal bath at a fixed temperature. Interactions among the particles themselves are neglected, and thus the equation is linear. Various versions of the linear Boltzmann equation are used to model phenomena such as neutron scattering [27,28], radiative transfer [1] and cometary flows [16] (we refer to [13, Chapter XXI] for a detailed presentation of the mathematical theory of linear collisional kinetic equations), and appears in some non-linear models as a background interaction term [4,10,17]. On the other hand, a technical motivation for our results is that inequalities relating the logarithmic entropy to its production rate are interesting by themselves, and are helpful in the study of non-linear models involving a linear Boltzmann term. These inequalities are intriguing and have been studied in [5] in the case of hard potentials; we intend to complete these ideas by looking at the case of soft potentials. Our strategy of proof is close to that in [11] (which applies to the non-linear Boltzmann equation), and is based on this type of inequalities.

The linear Boltzmann equation we consider here has been studied in several previous works [5,23,24,31]. Its spectral gap properties are understood since [19], with constructive estimates on the size of the spectral gap in $L^2(M^{-1})$ (where M is the equilibrium) for hard potentials given in [24]. Semigroup techniques were used in [23,27] to obtain convergence to equilibrium for all initial conditions in L^1, without explicit rates. An important related equation is the *linearised* Boltzmann equation, which has been treated for example in [3,8,22,29,30]. Roughly speaking, the spectral gap properties of both equations (linear and linearised) are now understood in a variety of spaces. The difference in our present approach is that it is based on functional inequalities for the logarithmic entropy, which have their own interest and are more robust when applied to models with mixed linear and non-linear terms [4,10].

Similar questions for the non-linear space-homogeneous Boltzmann equation have also been considered in the literature, and we refer to [15] for an overview and to [11] for convergence results with soft potentials. Mathematical questions are more involved in the non-linear setting, and of course the picture becomes more complete in the linear case. However, the question remains open regarding the validity of some functional inequalities in the non-cutoff case; we comment on this at the end of this introduction.

1.1. The linear Boltzmann operator

In this work we will be interested in properties of the solution to the following spatially homogeneous Boltzmann equation

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