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Preduals of spaces of holomorphic functions and the approximation property

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A R T I C L E I N F O

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Dedicated to the memory of Professor Jorge Mujica (1946–2017)

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A B S T R A C T

Let U be a balanced open subset of a Hausdorff complex locally convex space E. In this paper we represent the predual $G(U)$ of the space of holomorphic functions as the projective limit of the preduals of spaces of holomorphic functions that are bounded on certain subsets of U. As an application we prove that if E is a Banach space, then it has the approximation property if and only if $G(U)$ has the approximation property. We also give a new proof of a result due to Aron and Schottenloher stating that if E is a complex Banach space with the approximation property then the space of holomorphic functions $\mathcal{H}(U)$, with the compact-open topology, has the approximation property.

1. Introduction

Let E be a Hausdorff complex locally convex space. Let U be a non-empty open subset of E. In 1976, Mazet [15] proved the existence of a complete locally convex space $G(U)$ such that $G(U)' = \mathcal{H}(U)$, where $\mathcal{H}(U)$ denotes the space of holomorphic functions from U into $\mathbb{C}$. The space $G(U)$ is called the predual of $\mathcal{H}(U)$. Mujica and Nachbin [18, Theorem 2.1] gave a new proof of this result and the following description of $G(U)$:

$$G(U) = \{\phi \in \mathcal{H}(U)^* : \phi|_{B_{\mathcal{U}}^{\alpha}} \text{ is } \tau_0\text{-continuous for every } \alpha, \mathcal{U}\},$$

where $\alpha = (\alpha_n)_{n=1}^{\infty}$ are sequences of positive real numbers, $\mathcal{U} = (U_n)_{n=1}^{\infty}$ are the countable increasing open covers of U, and τ_0 denotes the compact-open topology on $\mathcal{H}(U)$. The space $G(U)$ is endowed with the topology of uniform convergence on all sets

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$$B_{\mathcal{U}}^{\alpha} = \{f \in \mathcal{H}^{\infty}(\mathcal{U}) : \|f\|_{U_n} \leq \alpha_n \text{ for every } n \in \mathbb{N}\},$$

where

$$\|f\|_{U_n} = \sup_{x \in U_n} |f(x)|,$$

and $\mathcal{H}^{\infty}(\mathcal{U})$ denotes the Fréchet space

$$\mathcal{H}^{\infty}(\mathcal{U}) = \{f \in \mathcal{H}(U) : f(U_n) \text{ is bounded in } \mathbb{C} \text{ for every } n \in \mathbb{N}\},$$

endowed with the topology of uniform convergence on all sets U_n.

Mujica and Nachbin also proved that $G(U)'_i = (\mathcal{H}(U), \tau_\delta)$, where $G(U)'_i$ denotes the inductive dual of $G(U)$ and τ_δ the locally convex inductive limit topology on $\mathcal{H}(U)$, defined by

$$(\mathcal{H}(U), \tau_\delta) = \text{ind}_{\mathcal{U}} \mathcal{H}^{\infty}(\mathcal{U}),$$

where $\mathcal{U}$ varies among the countable increasing open covers of U. This topology was introduced independently by Coeuré [7] and Nachbin [19], and the fact that $G(U)'_i = (\mathcal{H}(U), \tau_\delta)$ allow us to obtain topological properties of $\mathcal{H}(U)$ from topological properties of $G(U)$.

Mujica [16, Theorem 2.1] also constructed the predual $G^{\infty}(\mathcal{U})$ of $\mathcal{H}^{\infty}(\mathcal{U})$ which is the complete Hausdorff locally convex space given by

$$G^{\infty}(\mathcal{U}) = \{\psi \in \mathcal{H}^{\infty}(\mathcal{U})' : \ \psi|_{B_{\mathcal{U}}^{\alpha}} \text{ is } \tau_0\text{-continuous for every } \alpha\},$$

endowed with the topology of uniform convergence on all the sets $B_{\mathcal{U}}^{\alpha}$, where α runs over the sequences of positive numbers.

Several authors have obtained topological properties of $G(U)$ (see [3–6,10,18]). For example, Boyd [5, Theorem 14] proved that if U is a balanced open subset of a Fréchet–Montel space E such that $G(E)$ is Montel, then E has the approximation property if and only if $G(U)$ has the approximation property. Another result of this type is due to Çaliskan [6, Proposition 6.6], who proved that if U is a balanced open subset of a Fréchet space E, then E has the compact approximation property if and only if $G(U)$ has the compact approximation property.

In this paper we represent the predual $G(U)$ of the space of holomorphic functions as the projective limit of the spaces $G^{\infty}(\mathcal{U})$ (see Theorem 2.3), when $\mathcal{U}$ runs over all countable increasing open covers of U. As application of this result we prove that if U is a balanced open subset of a Banach space E, then E has the approximation property if and only if $G(U)$ has the approximation property (the definition of approximation property in locally convex spaces is given in the beginning of Section 3). We also give a new proof of Aron and Schottenloher [1, Theorem 2.2] which states that if U is a balanced open subset of a Banach space E and E has the approximation property, then $(\mathcal{H}(U), \tau_0)$ has the approximation property.

It is worth to mention that a Banach space is Montel if and only if it has finite dimension. Consequently the class of spaces studied by Boyd [5, Theorem 14] and the class studied by us are, in general, different.

It is well known that the approximation property implies the compact approximation property, but the converse is in general false (see [21]).

We refer the reader to [8,17] for further reading on infinite-dimensional holomorphy, to [13] for a detailed study on projective and inductive limits, and to [2,9] for the properties of the inductive dual.

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