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Well-posedness for stochastic scalar conservation laws with the initial-boundary condition

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A B S T R A C T

In this paper, we are interested in the initial-(non-homogeneous) Dirichlet boundary value problem for a multi-dimensional scalar non-linear conservation law with a multiplicative stochastic forcing. We introduce a notion of "renormalized" kinetic formulations in which the kinetic defect measures on the boundary of a domain are truncated. In such a kinetic formulation we establish a result of well-posedness of the initial-boundary value problem under only the assumptions (H_1), (H_2) and (H_3) stated below, which are very similar ones in [6].

1. Introduction

In this paper, we consider the first-order stochastic conservation laws of the following type

$$du + \operatorname{div}(A(u))dt = \Phi(u)dW(t) \quad \text{in} \quad \Omega \times Q, \tag{1.1}$$

with the initial condition

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in} \quad \Omega \times D, \tag{1.2}$$

and the formal boundary condition

$$\text{“}u = u_b\text{”} \quad \text{on} \quad \Omega \times \Sigma. \tag{1.3}$$

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Here $D \subset \mathbb{R}^d$ is a bounded convex domain with a Lipschitz boundary ∂D, $T > 0$, $Q = D \times (0,T)$, $\Sigma = \partial D \times (0,T)$ and W is a cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. More precisely, $(\mathscr{F}_t)$ is a complete right-continuous filtration and $W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k$ with $(\beta_k)_{k\geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)$ and $(e_k)_{k\geq 1}$ a complete orthonormal system in a separable Hilbert space H (cf. [4] for example). Our purpose of this paper is to present a well-posedness result for initial-boundary value problem (1.1)–(1.3) under the hypothesis (H_1), (H_2) and (H_3) stated below, which are very similar ones in the pioneering paper [6].

In the deterministic case of $\Phi = 0$, the problem has been extensively studied. It is well-known that a smooth solution is constant along characteristic curves, which can intersect each other and shocks can occur. Moreover, when the characteristic intersects both $\{0\} \times D$ and Σ, the problem (1.1)–(1.3) would be overdetermined if (1.3) were assumed in the usual sense. Thus, an appropriate framework of entropy solutions, together with entropy-boundary conditions, has been considered to obtain the well-posedness of (1.1)–(1.3) with $\Phi = 0$. Bardos, Le Roux and Nédélec [1] first gave an interpretation of the boundary condition (1.3) as an "entropy" inequality on Σ, which is the so-called BLN condition, and proved the well-posedness of (1.1)–(1.3) with $\Phi = 0$. However, their result requires the existence of trace on Σ with respect to L^1 strong topology, and so they had to consider solutions in the BV setting. Otto [19] has extended their result to the L^∞ setting by introducing the notion of boundary entropy flux pairs. On the other hand, Imbert and Vovelle [12] gave a kinetic formulation to (1.1)–(1.3) with $\Phi = 0$ and proved the uniqueness of kinetic solutions in the L^∞ space. Concerning the initial-boundary value problem for deterministic degenerate parabolic equations, see [18] and [14].

It is natural for applications to consider a conservation law with a stochastic forcing $\Phi(u)dW(t)$ which appears in wide variety of fields as physics, engineering and others. As regard the Cauchy problem for the stochastic conservation law (1.1) on $\mathbb{R}^d$ it has been studied in [13] in the case of additive noise, in [9] in the case of multiplicative noise, where the uniqueness of the "strong" entropy solution is established in any dimension, but the existence in one dimension. For the existence in any dimension see [3]. The Cauchy problem for (1.1) with a general multiplicative noise $\Phi(u)dW(t)$ in a d-dimensional torus has been studied in [6], in which Debussche and Vovelle proved the well-posedness of (1.1) by using a kinetic formulation. The main advantage in using kinetic formulations developed by Lions, Perthame and Tadmor for the deterministic case [17] is that the formulation keeps track of the dissipation of noise by solutions as well as it works in the L^1 setting. Those results have been extended to the case of degenerated parabolic stochastic equations in [5] and [16].

There are several papers concerning the initial boundary value problem for stochastic conservation laws. Vallet and Wittobold [21] extended the result of Kim [13] to the d-dimensional Dirichlet problem with additive noise, and then Bauzet, Vallet and Wittbold [2] studied the Dirichlet problem in the case of multiplicative noise. In [21] and [2] it is assumed that the flux function A is global Lipschitz and the Dirichlet boundary datum is zero. The homogeneous boundary condition is formulated in the sense of Carrillo, which formulates the semi-Kružkov entropies.

In the recent paper [15] Kobayasi and Noboriguchi investigated the non-homogeneous Dirichlet boundary problem (1.1)–(1.3) under the hypothesis (H_1), (H_2) and (H_3). The hypothesis (H_1) implies that the flux function A is not always Lipschitz but locally Lipshitz, and hence an important example of inviscid Burgers' equation can be included. The basic idea of the arguments in [15] is analogous to that of [6] and [12], but the stochastic forcing case is significantly different from the deterministic case. A "stochastic kinetic solution" u might blow up at the boundary ∂D even if the data u_0, u_b in (1.2), (1.3) are bounded. As we see in [12], the defect measure $\bar{m}^\pm$ on the boundary $\Sigma \times \mathbb{R}_\xi$ play an important role. In particular, it is crucial that $\bar{m}^+$ (resp. $\bar{m}^-$) vanishes for $\xi >> 1$ (resp. $\xi << -1$) in the proof of uniqueness. These properties for $\bar{m}^+$, $\bar{m}^-$ come from the boundedness of weak kinetic (entropy) solutions. To the contrary, in the stochastic forcing case we have no pathwise L^∞ estimate of kinetic (entropy) solution $u(t)$ even though both of initial datum u_0 and boundary datum u_b belong to L^∞: It is known only that $\mathbb{E} \sup_{0\leq t\leq T} \|u(t)\|^p_{L^p(D)}$ is finite for