J. Math. Anal. Appl. 462 (2018) 982–998

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A Kato's second type representation theorem for solvable sesquilinear forms

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ARTICLE INFO

Article history:
Received 1 August 2017
Available online 27 December 2017
Submitted by A. Lunardi

Keywords:
Kato's representation theorems
q-closed/solvable sesquilinear forms
Radon–Nikodym-like representations

ABSTRACT

Kato's second representation theorem is generalized to solvable sesquilinear forms. These forms need not be non-negative nor symmetric. The representation considered holds for a subclass of solvable forms (called hyper-solvable), precisely for those whose domain is exactly the domain of the square root of the modulus of the associated operator. This condition always holds for closed semibounded forms, and it is also considered by several authors for symmetric sign-indefinite forms. As a consequence, a one-to-one correspondence between hyper-solvable forms and operators, which generalizes those already known, is established.

1. Introduction

A sesquilinear form Ω on a dense domain $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ is called *q-closed* if $\mathcal{D}$ can be made into a reflexive Banach space $\mathcal{D}[\|\cdot\|_\Omega]$, continuously embedded in $\mathcal{H}$, and such that the form is bounded in it. This allows to define a *Banach–Gelfand triplet* $\mathcal{D} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times$, where the arrows indicate continuous embeddings and $\mathcal{D}^\times$ is the conjugate dual space of $\mathcal{D}[\|\cdot\|_\Omega]$. We call Ω *solvable* if a perturbation of Ω with a bounded form Υ on $\mathcal{H}$, defines a bounded operator, with bounded inverse, which acts on the triplet (the set of these perturbations is denoted by $\mathfrak{P}(\Omega)$). These sesquilinear forms have been studied by Di Bella and Trapani in [1] and by Trapani and the author in [2].

As proved in [1], for a solvable sesquilinear form Ω there exists a closed operator T, with dense domain $D(T) \subseteq \mathcal{D}$, such that the following representation holds

$$\Omega(\xi, \eta) = \langle T\xi, \eta \rangle, \qquad \forall \xi \in D(T), \eta \in \mathcal{D}. \tag{1.1}$$

This extends the representation theorems for sesquilinear forms considered by many authors, as for instance by Kato [11], McIntosh [12], Fleige et al. [5–7], Grubišić et al. [8] and Schmitz [15] (for a more complete list see the references of [2]).

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https://doi.org/10.1016/j.jmaa.2017.12.058

For a non-negative closed form Ω, with positive associated operator T, Kato also proved the so-called *second representation theorem* [11, Theorem VI.2.23]: $\mathcal{D} = D(T^{\frac{1}{2}})$, where T is the operator appearing in (1.1), and

$$\Omega(\xi, \eta) = \langle T^{\frac{1}{2}}\xi, T^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D}.$$

In the case where Ω is a general sectorial closed form, Kato [10] generalized the representation as

$$\Omega(\xi, \eta) = \langle T^{\frac{1}{2}}\xi, T^{*\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D},$$

where $T^{\frac{1}{2}}$ and $T^{*\frac{1}{2}}$ are fractional powers of T and T^* (see [9]), respectively, under the assumption that $\mathcal{D} = D(T^{\frac{1}{2}}) = D(T^{*\frac{1}{2}})$. However, this latter condition does not always hold, as shown by McIntosh [13].

McIntosh [12], Fleige et al. [4,5], Grubišić et al. [8] and Schmitz [15] adapted the second representation theorem for symmetric sesquilinear forms they considered. More precisely, in [4,5,8,15] it is proved that, if $\mathcal{D} = D(|T|^{\frac{1}{2}})$ and $T = U|T|$ is the polar decomposition of T, then

$$\Omega(\xi, \eta) = \langle U|T|^{\frac{1}{2}}\xi, |T|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D}.$$

In this paper we adapt Kato's second representation theorem to a solvable sesquilinear form Ω (not necessarily symmetric), represented by an operator T, and with domain $\mathcal{D} = D(|T|^{\frac{1}{2}})$ (if this condition is satisfied then we say that Ω is *hyper-solvable*). It emerges that the condition $\mathcal{D} = D(|T|^{\frac{1}{2}}) = D(|T^*|^{\frac{1}{2}})$ is equivalent to $\mathcal{D} = D(|T|^{\frac{1}{2}})$ and we also prove, if Ω is hyper-solvable, that

$$\Omega(\xi, \eta) = \langle U|T|^{\frac{1}{2}}\xi, |T^*|^{\frac{1}{2}}\eta\rangle = \langle |T^*|^{\frac{1}{2}}U\xi, |T^*|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D} \tag{1.2}$$

where $T = U|T|$ is the polar decomposition of T.

The paper is organized as follows. In Section 2 we give a brief overview on q-closed and solvable forms, while in Section 3 we introduce a so-called *Radon–Nikodym-like representation* for a general q-closed/solvable form, i.e. an expression

$$\Omega(\xi, \eta) = \langle QH\xi, H\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D}, \tag{1.3}$$

where $Q \in \mathcal{B}(\mathcal{H})$ and H is a positive self-adjoint operator with $0 \in \rho(H)$ and domain $\mathcal{D}$. Moreover, we show that $\Upsilon \in \mathfrak{P}(\Omega)$ if, and only if, $Q + H^{-1}BH^{-1}$ is a bijection of $\mathcal{H}$, where B is the operator associated to Υ.

The sesquilinear forms studied in [8] are exactly defined as in (1.3) with Q symmetric. Following [8], we can give another expression of the operator T associated to Ω in (1.3). Indeed, the domain $D(T)$ of T is equal to $D(T) = \{\xi \in \mathcal{D} : QH\xi \in \mathcal{D}\}$ and $T = HQH$.

In Section 4 we prove the special Radon–Nikodym-like representation that holds for hyper-solvable forms

$$\Omega(\xi, \eta) = \langle V|T + B|^{\frac{1}{2}}\xi, |T + B|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D},$$

where B is the bounded operator associated to $\Upsilon \in \mathfrak{P}(\Omega)$ and $V \in \mathcal{B}(\mathcal{H})$. If, moreover, $0 \in \rho(T)$ then there exists a unique bijection $W \in \mathcal{B}(\mathcal{H})$ such that

$$\Omega(\xi, \eta) = \langle W|T|^{\frac{1}{2}}\xi, |T|^{\frac{1}{2}}\eta\rangle, \qquad \forall \xi, \eta \in \mathcal{D}.$$

We also prove (1.2) and with the aid of the Radon–Nikodym-like representation we adapt the criteria, contained in Theorem 3.2 and Lemma 3.6 of [8], to ensure that a solvable form is hyper-solvable.

In Section 5 we consider the problem of representation to the converse direction; that is, for an operator T with certain properties we construct a solvable sesquilinear form (which is in particular hyper-solvable) with