J. Math. Anal. Appl. 461 (2018) 1711–1732

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Localization and compactness of operators on Fock spaces

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ARTICLE INFO

Article history:
Received 27 October 2017
Available online 19 December 2017
Submitted by J. Xiao

Keywords:
Fock space
Weakly localized operator
Compactness

ABSTRACT

For $0 < p \le \infty$, let F^p_φ be the Fock space induced by a weight function φ satisfying $dd^c\varphi \simeq \omega_0$. In this paper, given $p \in (0,1]$ we introduce the concept of weakly localized operators on F^p_φ, we characterize the compact operators in the algebra generated by weakly localized operators. As an application, for $0 < p < \infty$ we prove that an operator T in the algebra generated by bounded Toeplitz operators with BMO symbols is compact on F^p_φ if and only if its Berezin transform satisfies certain vanishing property at ∞. In the classical Fock space, we extend the Axler–Zheng condition on linear operators T, which ensures T is compact on F^p_α for all possible $0 < p < \infty$.

1. Introduction

Let $H(\mathbb{C}^n)$ be the collection of all entire functions on $\mathbb{C}^n$, and let $\omega_0 = dd^c|z|^2$ be the Euclidean Kähler form on $\mathbb{C}^n$, where $d^c = \frac{\sqrt{-1}}{4}(\overline{\partial} - \partial)$. Set $B(z,r)$ to be the Euclidean ball in $\mathbb{C}^n$ with center z and radius r, and $B(z,r)^c = \mathbb{C}^n \backslash B(z,r)$. Throughout the paper, we assume that $\varphi \in C^2(\mathbb{C}^n)$ is real-valued and there are two positive numbers M_1, M_2 such that

$$M_1\omega_0 \le dd^c\varphi \le M_2\omega_0 \tag{1.1}$$

in the sense of currents. The expression (1.1) will be denoted as $dd^c\varphi \simeq \omega_0$. Given $0 < p < \infty$ and a positive Borel measure μ on $\mathbb{C}^n$, let $L^p_\varphi(\mu)$ be the space defined by

$$L^p_\varphi(\mu) = \left\{ f \text{ is } \mu\text{-measurable on } \mathbb{C}^n : f(\cdot)e^{-\varphi(\cdot)} \in L^p(\mathbb{C}^n, d\mu) \right\}.$$

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[1] This research is partially supported by the National Natural Science Foundation of China (11601149, 11771139, 11571105), Natural Science Foundation of Zhejiang Province (LY15A010014).
[2] Research supported in part by a National Science Foundation DMS grant #0955432.

https://doi.org/10.1016/j.jmaa.2017.12.046
0022-247X/

When $d\mu = dV$, the Lebesgue measure on $\mathbb{C}^n$, we write L^p_φ for $L^p_\varphi(\mu)$ and set

$$\|f\|_{p,\varphi} = \left(\int\limits_{\mathbb{C}^n} \left| f(z) e^{-\varphi(z)} \right|^p dV(z) \right)^{\frac{1}{p}}.$$

For $0 < p < \infty$ the Fock space F^p_φ is defined as $F^p_\varphi = L^p_\varphi \cap H(\mathbb{C}^n)$, and

$$F^\infty_\varphi = \left\{ f \in H(\mathbb{C}^n) : \|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{C}^n} |f(z)| \, e^{-\varphi(z)} < \infty \right\}.$$

F^p_φ is a Banach space with norm $\|\cdot\|_{p,\varphi}$ when $1 \le p \le \infty$ and F^p_φ is a Fréchet space with distance $\rho(f,g) = \|f-g\|^p_{p,\varphi}$ if $0 < p < 1$. The typical model of φ is $\varphi(z) = \frac{\alpha}{2}|z|^2$ with $\alpha > 0$, which induces the classical Fock space. For this particular special weight φ, F^p_φ and $\|\cdot\|_{p,\varphi}$ will be written as F^p_α and $\|\cdot\|_{p,\alpha}$, respectively. The space F^p_α has been studied by many authors, see [2,5,7,18–21] and the references therein. Another special case is with $\varphi(z) = \frac{\alpha}{2}|z|^2 - \frac{m}{2}\ln(A + |z|^2)$ with suitable $A > 0$, and then F^p_φ is the Fock–Sobolev space $F^{p,m}_\alpha$ studied in [3,4].

It is well-known that F^2_φ is a Hilbert space with inner product

$$\langle f, g\rangle_{F^2_\varphi} = \int\limits_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-2\varphi(z)} dV(z).$$

Given $z, w \in \mathbb{C}^n$, the reproducing kernel of F^2_φ will be denoted by $K_z(w) = K(w,z)$. We write $k_z = \frac{K_z}{\|K_z\|_{2,\varphi}}$ to denote the normalized reproducing kernel. Given some bounded linear operator T on F^p_φ, the Berezin transform of T is well defined as

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle_{F^2_\varphi},$$

since $Tk_z \in F^p_\varphi \subset F^\infty_\varphi$ and $k_z \in F^1_\varphi$. Set P to be the projection from L^2_φ to F^2_φ, that is

$$Pf(z) = \int\limits_{\mathbb{C}^n} f(w)K(z,w)e^{-2\varphi(w)} dV(w) \quad \text{for } f \in L^2_\varphi.$$

For a complex Borel measure μ on $\mathbb{C}^n$ and $f \in F^p_\varphi$, we define the Toeplitz operator T_μ to be

$$T_\mu f(z) = \int\limits_{\mathbb{C}^n} f(w)K(z,w)e^{-2\varphi(w)} d\mu(w).$$

If $d\mu = gdV$, for short, we will use T_g to stand for the induced Toeplitz operator and will use that $\widetilde{g} = \widetilde{T_g}$.

In the case of Fock spaces F^2_α, fixed g bounded on $\mathbb{C}^n$, $|\langle T_g k_z, k_w\rangle|$ as a function of (z,w) decays very fast off the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$, see [20, Proposition 4.1]. From this point of view, Xia and Zheng in [20] introduced the notion of "sufficiently localized" operators on F^2_α which include the algebra generated by Toeplitz operators with bounded symbols, and they proved that, if T is in the C^*-algebra generated by the class of sufficiently localized operators, T is compact on F^2_α if and only if its Berezin transform tends to zero when z goes to infinity. In [10], Isralowitz extended [20] to the generalized Fock space F^2_φ with $dd^c\varphi \simeq \omega_0$. Isralowitz, Mitkovski and the third author extended Xia and Zheng's idea further in [11] to what they called "weakly localized" operators on F^p_φ with $1 < p < \infty$. They showed that, if T is in the C^*-algebra generated by the class of weakly localized operators, T is compact on F^p_φ if and only if its Berezin transform shares