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General approach to microscopic-type sets

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ABSTRACT

A new concept of small subsets of the real line is presented. It is a generalization of different kinds of microscopic sets considered previously. We compare these sets with Lebesgue null sets, with sets of strong measure zero. With this approach we are able to get families of fractals – sets with fractional Hausdorff dimension.

The notion of microscopic sets on the real line was introduced in the beginning of 21-st century in the paper [2] by J. Appell, E. D'Aniello and M. Väth. Deeper studies of such sets can be found in [1], [5], [9], [10], [11], [13], [16]. A good general reference on this topic is [7].

Definition 1. A set $E \subset \mathbb{R}$ is microscopic if for each $\epsilon > 0$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \text{ and } \lambda(I_n) \leq \epsilon^n \text{ for } n \in \mathbb{N},$$

where λ stands here for the Lebesgue measure.

The family of all microscopic sets will be denoted by $\mathcal{M}$.

A special role in this definition is played by a geometric sequence. A question about the consequences of replacing this specific sequence with another one was raised by J. Appell, E. D'Aniello and M. Väth in [2] and some results concerning this problem have been published ([6], [4]). They were based on the following approach.

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Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of increasing functions $f_n : (0,1) \to (0,1)$, such that $\lim_{x\to 0^+} f_n(x) = 0$ and there exists $x_0 \in (0,1)$ such that for every $x \in (0, x_0)$ the series $\sum_{n\in\mathbb{N}} f_n(x)$ is convergent and the sequence $(f_n(x))_{n\in\mathbb{N}}$ is nonincreasing. We denote by $\mathcal{H}$ the family of all sequences of functions with the properties described above.

Definition 2. A set $E \subset \mathbb{R}$ belongs to $\mathcal{M}_{(f_n)}$ if for each $x \in (0,1)$ there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \text{ and } \lambda(I_n) \le f_n(x) \text{ for } n \in \mathbb{N}.$$

Observe that for the sequence of functions $f_n(x) := x^n,\ n \in \mathbb{N}$, we have $\mathcal{M}_{(f_n)} = \mathcal{M}$.

The definition of the family $\mathcal{M}_{(f_n)}$ has an equivalent "countable" version:

$\mathcal{M}_{(f_n)} = \left\{E \subset \mathbb{R} :\ \forall_{k\in\mathbb{N}}\ \exists_{\{I_n\}_{n\in\mathbb{N}}}\ \left(E \subset \bigcup_{n\in\mathbb{N}} I_n\ \wedge\ \forall_{n\in\mathbb{N}}\ \lambda(I_n) \le f_n\left(\frac{1}{k}\right)\right)\right\}$.

Lately another generalization of microscopic sets has been considered.

Let Ω denote the family of all non-increasing sequences $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers such that the series $\sum_{n\in\mathbb{N}} a_n$ is convergent.

Let $\{a_n\}_{n\in\mathbb{N}} \in \Omega$.

Definition 3. (see [8]) A set $E \subset \mathbb{R}$ belongs to $\mathcal{M}_{\{a_n\}}$ if for each $k \in \mathbb{N}$ there exists a sequence of intervals $\{I_n^{(k)}\}_{n\in\mathbb{N}}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n^{(k)} \quad \text{and} \quad \forall_{n\in\mathbb{N}}\ \lambda(I_n^{(k)}) \le a_{n\cdot k}.$$

For $a_n := \varepsilon^n,\ n \in \mathbb{N}$, where $\varepsilon \in (0,1)$ we have $\mathcal{M}_{\{a_n\}} = \mathcal{M}$.

Therefore we have two generalizations of microscopic sets. It turned out that a version with a sequence of functions is more general than this one with a sequence of numbers.

Theorem 4. *For every sequence of numbers $\{a_n\} \in \Omega$ there exists a sequence of functions $(f_n) \in \mathcal{H}$ such that*

$$\mathcal{M}_{\{a_n\}} = \mathcal{M}_{(f_n)}.$$

Proof. It is enough to define $f_n(\frac{1}{k}) := a_{n\cdot k}$ for $k \in \mathbb{N}$, and to define f_n on the intervals $(\frac{1}{k+1}, \frac{1}{k})$ so to make them continuous and increasing, and also making sure that for any $x \in (0,1)$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is nonincreasing. □

Theorem 5. *There are sequences of functions $(f_n) \in \mathcal{H}$ such that for every sequence of numbers $\{a_n\} \in \Omega$*

$$\mathcal{M}_{\{a_n\}} \neq \mathcal{M}_{(f_n)}.$$

Proof. The proof is immediate if we consider properties of the families $\mathcal{M}_{\{a_n\}}$ and $\mathcal{M}_{(f_n)}$. For every sequence of numbers $\{a_n\} \in \Omega$ the family $\mathcal{M}_{\{a_n\}}$ is a σ-ideal (see [8]), whereas there are sequences of functions $(f_n) \in \mathcal{H}$ such that $\mathcal{M}_{(f_n)}$ is not even an ideal (see [4]). □

It seems to be interesting to give a characterization of sequences of functions $(f_n) \in \mathcal{H}$ for which a sequence of numbers $\{a_n\} \in \Omega$ such that $\mathcal{M}_{\{a_n\}} = \mathcal{M}_{(f_n)}$ can be found.

We have only a sufficient condition here.