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Non-uniform dependence for the periodic higher dimensional Camassa–Holm equations ☆

Yongye Zhao [a,*], Meiling Yang [b], Yongsheng Li [b]

[a] Department of Basic Courses, Guangzhou Maritime University, Guangzhou, Guangdong 510725, PR China
[b] School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, PR China

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ABSTRACT

In this paper, we investigate the dependence on initial data of solutions to the higher dimensional Camassa–Holm equations with periodic boundary condition in Besov spaces. We show that when $s > 1 + \frac{d}{2}$ $(d \geq 2)$ and $1 \leq r \leq \infty$, the solution map is not uniformly continuous from $B^s_{2,r}(\mathbb{T}^d)$ into $C\left([0,T]; B^s_{2,r}(\mathbb{T}^d)\right)$ for $r < \infty$ or from $B^s_{2,\infty}(\mathbb{T}^d)$ into $L^\infty\left([0,T]; B^s_{2,\infty}(\mathbb{T}^d)\right)$ for $r = \infty$.

1. Introduction

In this paper, we study the higher dimensional Camassa–Holm equations with periodic boundary condition,

$$\begin{cases} m_t + u \cdot \nabla m + \nabla u^T \cdot m + m(\text{div } u) = 0 & t \geq 0,\ x \in \mathbb{T}^d, \\ u(0,x) = u_0(x) & x \in \mathbb{T}^d. \end{cases} \tag{1.1}$$

Here $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ is the torus. The vector field $u = (u_1(t,x), \cdots, u_d(t,x))$ is the velocity of the fluid, $m = (\mathrm{I} - \Delta)u$ is the momentum of the fluid, and $u = G * m$, where G is the Green function for the Helmholtz operator $\mathrm{I} - \Delta$.

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* Corresponding author.
E-mail addresses: yongyezhao@163.com (Y. Zhao), ymling2010@126.com (M. Yang), yshli@scut.edu.cn (Y. Li).

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Notice that (1.1) can be viewed as higher dimensional generalization of the one dimensional Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m = 0, \qquad m = u - u_{xx}. \tag{1.2}$$

It was shown in [4] that the CH equation is bi-Hamiltonian, and completely integrable, and thus admits an infinite number of conservation laws. The integrability of the CH equation (as an infinite-dimensional Hamiltonian system) was studied in [12,18,24]. Particularly, a remarkable feature of the CH equation is the presence of wave breaking, i.e. the solution remains bounded but its slope becomes unbounded in finite time [8]. Another remarkable phenomenon of the CH equation is the presence of the peaked solitons solutions [4,11]. The initial value problem and initial boundary value problem for CH equation have been analyzed in [8,9,14,15,17]. It has been shown that the CH equation is locally well-posed (see [8,15,25,35]). Moreover, it has global strong solutions [6,8] and weak solutions [13,33]. The existence of the global conservative and dissipative solutions to the CH equation were studied in [2,3]. The essential feature of the CH equation revealed in recent papers [7,10,11] is that the traveling waves with a peak at their crest is exactly like the waves of greatest height solutions to the governing equations for water waves.

As for (1.1), it was proposed exactly in the way that a class of its singular solutions generalize the peakon solutions of the CH equation to higher spatial dimensions [22]. It was also regarded as Euler–Poincaré equations associated with the diffeomorphism group (EPDiff equations) in [21], and was shown that the last three terms on the left-hand side of (1.1) model convection, stretching and expansion of a fluid with the velocity u and the momentum m, respectively. The local well-posedness, blow up criteria, global and blow-up solutions of the Cauchy problem for (1.1) have been discussed in [5,26,34].

In our situation, we are going to investigate the continuity of the solution map defined by the problem (1.1). Actually, the issue of continuity properties of the solution map has been the subject of many papers (see [20,19,23,28,29,31,30]) and the references therein. For example, the issue of non-uniform dependence on initial data for the CH equation in $H^s(\mathbb{T})$ and Besov space $B^s_{p,r}(\mathbb{T})$ was studied in [20] and [30] respectively. For the non-uniform dependence in critical Besov space, we refer to [29,31]. As for the multi-dimensional case, the non-uniform dependence for the following Euler equations

$$\begin{cases} u_t + u\cdot\nabla u + \nabla P = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x), x\in\Omega, t\in\mathbb{R}, \end{cases} \tag{1.3}$$

was firstly obtained in [19], where Ω is a d-dimensional domain, $d \ge 2$. In [19], Himonas and Misiołek proved that the solution map for (1.3) is not uniformly continuous from $H^s(\Omega)$ into $C([0,T];H^s(\Omega))$ for any $s\in\mathbb{R}$ if $\Omega = \mathbb{T}^d$ and for any $s > 0$ if $\Omega = \mathbb{R}^d$. Recently, Tang and Liu [28] extended partially the above results. They proved that for any $s \in \mathbb{R}$ and $1 \le r \le \infty$, the solution map of (1.3) is not globally uniformly continuous in $B^s_{2,r}(\mathbb{T}^d)$.

However, since the local existence results for (1.1) have been established in Besov spaces (see [34]), it is reasonable to ask whether some results similar to these obtained in [28,30] holds for the higher dimensional CH equation (1.1) in the Besov spaces. In this paper, we will show that for (1.1), the dependence on initial data is optimal in the sense that the solution map is not uniformly continuous in $B^s_{2,r}(\mathbb{T}^d)$. Compared with the CH equation in one dimension, more estimates in higher dimensions are required in this paper. To accomplish our purpose, we shall rewrite (1.1) into the following form (see [34] for the details):

$$\begin{cases} u_t + u\cdot\nabla u = F(u(t,x)), & t\in\mathbb{R}^+, \quad x\in\mathbb{T}^d, \\ u(0,x) = u_0(x), & x\in\mathbb{T}^d, \end{cases} \tag{1.4}$$