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Spectrum of the Iwatsuka Hamiltonian at thresholds

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A B S T R A C T

We consider the bi-dimensional Schrödinger operator with unidirectionally constant magnetic field, H_0, sometimes known as the "Iwatsuka Hamiltonian". This operator is analytically fibered, with band functions converging to finite limits at infinity. We first obtain the asymptotic behavior of the band functions and its derivatives. Using this results we give estimates on the current and on the localization of states whose energy value is close to a given *threshold* in the spectrum of H_0. In addition, for non-negative electric perturbations V we study the spectral properties of $H_0 \pm V$, by considering the Spectral Shift Function associated to the operator pair $(H_0 \pm V, H_0)$. We compute the asymptotic behavior of the Spectral Shift Function at the thresholds, which are the only points where it can grow to infinity.

1. Introduction

1.1. Context and motivation

Let b be a scalar magnetic field in $\mathbb{R}^2$ that is translationally invariant in the sense that it does not depend on one of the spatial variables: $b(x, y) = b(x)$. Let $a : \mathbb{R} \to \mathbb{R}$ be the function $a(x) = \int_0^x b(t)\mathrm{d}t$. Then the potential $A(x) := (0, a(x))$ satisfies $\operatorname{curl} A = b$. In this article we study the magnetic Schrödinger operator

$$H_0 := (-i\nabla - A)^2 = -\partial_x^2 + (-i\partial_y - a(x))^2$$

acting in $L^2(\mathbb{R}^2)$.

In [20] Akira Iwatsuka considered this class of Hamiltonians in order to show new examples of magnetic Schrödinger operators with purely absolutely continuous spectrum. This model has many interesting properties, and some of them have been well studied in the past. For instance, in the physics literature it

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can be seen as an open quantum waveguide in the sense that the presence of such magnetic field generates transport for a spinless particle in a plane (see [25] for references). Rigorous propagation properties of this model are described for example in [23,22,10,19]. Additionally, the spectral properties of H_0 have also been studied. In this sense, one important conjecture on the spectrum of H_0 is that it should be purely absolutely continuous as long as b is non-constant. However, although it have been found different conditions on b that ensure this property, the general result remains unproved (see [20,23,14,32]).

In order to describe the problems that we propose to study in this article it is necessary to write a well known decomposition of the operator H_0.

Fiber decomposition of H_0. Let $\mathcal{F}$ be the partial Fourier transform with respect to the translationally invariant variable $y \in \mathbb{R}$:

$$(\mathcal{F}u)(x,k) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-iky} u(x,y)\,\mathrm{d}y, \quad \text{for } u \in C_0^\infty(\mathbb{R}^2).$$

Then

$$\mathcal{F} H_0 \mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h(k)\,\mathrm{d}k, \tag{1}$$

where $h(k)$ is a self-adjoint operator acting in $L^2(\mathbb{R})$, defined by

$$h(k) = -\frac{d^2}{\mathrm{d}x^2} + (a(x) - k)^2, \quad k \in \mathbb{R}. \tag{2}$$

For any $k \in \mathbb{R}$, under hypotheses (3) below, $h(k)$ has compact resolvent, therefore its spectrum is discrete, and moreover, it is simple. We denote the increasing sequence of eigenvalues by $\{E_n(k)\}_{n=1}^\infty$. For any $n \in \mathbb{N}$, the *band function* $E_n(\cdot)$ is analytic as a function of $k \in \mathbb{R}$ [20].

In a broad sense, the results of this article are valid for magnetic fields that satisfy:

$$\begin{aligned} &\text{a)} \quad b \in C^\infty(\mathbb{R}). \\ &\text{b)} \quad b \text{ is increasing.} \\ &\text{c)} \quad \lim_{x\to\pm\infty} b(x) = b_\pm, \quad \text{for } b_+ > b_- > 0. \end{aligned} \tag{3}$$

Even more, each one of these hypotheses can be relaxed, but the results are much easier to read for these kind of magnetic fields.

Set $\underline{\mathcal{E}}_n := b_-(2n-1)$ and $\overline{\mathcal{E}}_n := b_+(2n-1)$, then, condition (3) implies that $\underline{\mathcal{E}}_n \le E_n(k) \le \overline{\mathcal{E}}_n$ for all $k \in \mathbb{R}$. Furthermore, $E_n(\cdot)$ is strictly increasing for any $n \in \mathbb{N}$, and $\lim_{k\to+\infty} E_n(k) = \overline{\mathcal{E}}_n$; $\lim_{k\to-\infty} E_n(k) = \underline{\mathcal{E}}_n$ for all $n \in \mathbb{N}$, see [20,23]. In particular, the spectrum of H_0, $\sigma(H_0)$, is purely absolutely continuous and

$$\sigma(H_0) = \bigcup_{n\ge1} \overline{E_n(\mathbb{R})} = \bigcup_{n\ge1} [\underline{\mathcal{E}}_n, \overline{\mathcal{E}}_n]. \tag{4}$$

Moreover, using decomposition (1) and the monotonicity of E_n it is possible to see that the multiplicity of the spectrum of H_0 changes at any point in the set of points $\{\underline{\mathcal{E}}_n, \overline{\mathcal{E}}_n\}_{n=1}^\infty$. This set of points will be referred as *thresholds* in $\sigma(H_0)$. We denote it by $\mathcal{T}_{H_0}$.

1.2. Description of the main results of the article

The first step in the physical description of our magnetic system at energies near thresholds, requires to precise the behavior of the band functions when $k \to \pm\infty$. Therefore, our first task in this article is to