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Some elliptic problems with singular nonlinearity and advection for Riemannian manifolds ☆

João Marcos do Ó [a,*], Rodrigo G. Clemente [b]

[a] *Department of Mathematics, Federal University of Paraíba, 58051-900, João Pessoa, PB, Brazil*
[b] *Department of Mathematics, Rural Federal University of Pernambuco, 52171-900, Recife, Pernambuco, Brazil*

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A B S T R A C T

We are interested in regularity properties of semi-stable solutions for a class of singular semilinear elliptic problems with advection term defined on a smooth bounded domain of a complete Riemannian manifold with zero Dirichlet boundary condition. We prove uniform Lebesgue estimates and we determine the critical dimensions for these problems with nonlinearities of the type Gelfand, MEMS and power case. As an application, we show that extremal solutions are classical whenever the dimension of the manifold is below the critical dimension of the associated problem. Moreover, we analyze the branch of minimal solutions and we prove multiplicity results when the parameter is close to critical threshold and we obtain uniqueness on it. Furthermore, for the case of Riemannian models we study properties of radial symmetry and monotonicity for semi-stable solutions.
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1. Introduction

Let $(\mathcal{M}, g)$ be a complete Riemannian manifold with dimension N, $\Omega \subset \mathcal{M}$ a smooth bounded domain and $A(x)$ a smooth vector field over $\overline{\Omega}$. In the present paper, we investigate the following class of nonlinear elliptic differential equations involving singular nonlinearities and advection

$$\begin{cases} -\Delta_g u + A(x) \cdot \nabla_g u = \lambda f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{P_λ}$$

We analyze (P_λ) for the following types of nonlinearities:

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* Corresponding author.
E-mail addresses: jmbo@pq.cnpq.br (J.M. do Ó), rodrigo.clemente@ufrpe.br (R.G. Clemente).

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$$
\begin{array}{llll}
(i) & f(s) = e^s & \text{(Gelfand)} \\
(ii) & f(s) = (1+s)^m,\ m > 1 & \text{(Power-type)} \\
(iii) & f(s) = 1/(1-s)^2 & \text{(MEMS)}
\end{array} \tag{1.1}
$$

The main purpose of this paper is to study the minimal branch and regularity properties for minimal solutions of (P_λ). We first prove that there exists some positive finite critical parameter λ^* such that for all $0 < \lambda < \lambda^*$ the problem (P_λ) has a smooth minimal stable solution $\underline{u}_\lambda$ while for $\lambda > \lambda^*$ there are no solutions of (P_λ) in any sense (cf. Theorems 1.1). We determine the critical dimension N^* for this class of problems, precisely we prove that the extremal solution of (P_λ) is regular for $N \leq N^*$ and it is singular for $N > N^*$. We see that the critical dimension depends only on the nonlinearity $f(s)$ and does not depend of the Riemannian manifold $\mathcal{M}$ (cf. Theorem 1.2 and (1.4)). For that, we establish L^∞ estimates, which are crucial in our argument to obtain regularity of the extremal solutions. We also prove multiplicity of solutions near the extremal parameter and uniqueness on it (cf. Theorem 1.3 and Theorem 1.4). Moreover, we prove radial symmetry and monotonicity for semi-stable solutions of (P_λ) if $\Omega = \mathcal{B}_R$ is a geodesic ball of a Riemannian model $\mathcal{M}$ (cf. Theorem 1.5).

1.1. Statement of main results

Before we state our main results we recall some standard notations and definitions related with problem (P_λ). Next we are assuming the following values for s_0, which depends of the type of considered nonlinearity, precisely,

$$
\begin{array}{lllll}
(i) & s_0 = +\infty & \text{if} & f(s) = e^s & \text{(Gelfand)} \\
(ii) & s_0 = +\infty & \text{if} & f(s) = (1+s)^m & \text{(Power-type)} \\
(iii) & s_0 = 1 & \text{if} & f(s) = 1/(1-s)^2 & \text{(MEMS)}
\end{array}
$$

Classical solution: $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution of (P_λ) if it solves (P_λ) in the classical sense (i.e. using the classical notion of derivative).

Weak solution: $u \in W_0^{1,2}(\Omega)$ is a weak solution of (P_λ) if $0 \leq u < s_0$ almost everywhere in Ω and $u = s_0$ in a subset with measure zero such that $f(u) \in L^2(\Omega)$ and

$$
\int_\Omega (\nabla_g u \cdot \nabla_g \phi + \phi A \cdot \nabla_g u)\, \mathrm{d}v_g = \lambda \int_\Omega f(u)\phi\, \mathrm{d}v_g, \quad \forall \phi \in W_0^{1,2}(\Omega). \tag{1.2}
$$

We also consider *weak subsolution* (*weak supersolution*) in analogy with this definition. For instance, $u \in W_0^{1,2}(\Omega)$ is a weak subsolution of (P_λ) if $0 \leq u < s_0$ almost everywhere in Ω and $u = s_0$ in a subset with measure zero such that $f(u) \in L^2(\Omega)$ with "$\leq$" ("$\geq$") instead of "$=$" in (1.2).

Minimal solution: For problem (P_λ), we say that a weak solution $u \in W_0^{1,2}(\Omega)$ is a minimal solution if $u \leq v$ almost everywhere for all v supersolution. We denote minimal solution of (P_λ) by $\underline{u}_\lambda$.

Regular solution: We say that a weak solution u of (P_λ) is a regular solution if $\sup_\Omega u < s_0$.

Semi-stable solution: We say that a classical solution u of (P_λ) is semi-stable solution provided that

$$
\int_\Omega \left(|\nabla_g \xi|^2 + \xi A(x) \cdot \nabla_g \xi\right) \mathrm{d}v_g \geq \int_\Omega \lambda f'(u)\xi^2 \mathrm{d}v_g, \quad \forall \xi \in C_0^1(\Omega). \tag{1.3}
$$

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