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Simultaneous bifurcation of limit cycles from a cubic piecewise center with two period annuli

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A B S T R A C T

We study the number of periodic orbits that bifurcate from a cubic polynomial vector field having two period annuli via piecewise perturbations. The cubic planar system $(x', y') = (-y((x-1)^2 + y^2), x((x-1)^2 + y^2))$ has simultaneously a center at the origin and at infinity. We study, up to first order averaging analysis, the bifurcation of periodic orbits from the two period annuli, first separately and second simultaneously. This problem is a generalization of [24] to the piecewise systems class. When the polynomial perturbation has degree n, we prove that the inner and outer Abelian integrals are rational functions and we provide an upper bound for the number of zeros. When the perturbation is cubic, the same degree as the unperturbed vector field, the maximum number of limit cycles, up to first order perturbation, from the inner and outer annuli is 9 and 8, respectively. When the simultaneous bifurcation problem is considered, 12 limit cycles exist. These limit cycles appear in three types of configurations: $(9, 3)$, $(6, 6)$ and $(4, 8)$. In the non-piecewise scenario, only 5 limit cycles were found.

1. Introduction

The knowledge of the existence of periodic solutions is very important for understanding the dynamics of differential systems. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the mechanism. The first formalization of this procedure was given by Fatou in 1928, see [8]. Nevertheless, Buica and Llibre [1] extended the averaging theory for studying periodic orbits to continuous differential systems using mainly the Brouwer degree theory. Recently, the averaging theory for studying periodic orbits to piecewise differential systems has been developed, see [16,17] for example. Here we use the same approach as [2].

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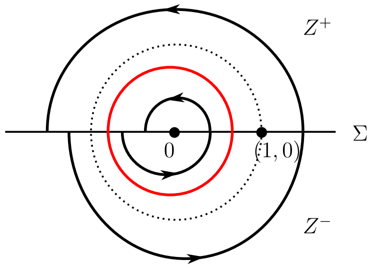

Fig. 1. A possible phase portrait of system (1).

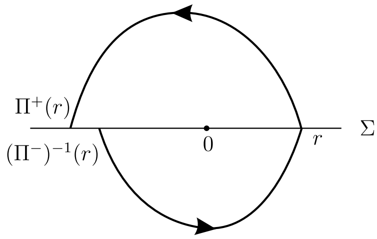

Fig. 2. Return map for system (1).

Consider the perturbed polynomial piecewise differential system

$$Z^{\pm} = \begin{cases} \dot{x} = -y((x-1)^2 + y^2) + \varepsilon P_n^{\pm}(x,y), \\ \dot{y} = x((x-1)^2 + y^2) + \varepsilon Q_n^{\pm}(x,y), \end{cases} \quad \text{if } (x,y) \in \Sigma^{\pm}, \tag{1}$$

with $P_n^{\pm}$ and $Q_n^{\pm}$ polynomials of degree n and $\Sigma^{\pm} = \{(x,y) : \pm y > 0\}$. An example of the phase portrait of the above system, for ε small, is drawn in Fig. 1.

Following [2], the limit cycles of (1) correspond to the zeros of the difference map $\Pi^+(r) - (\Pi^-)^{-1}(r)$, see Fig. 2. Moreover, for ε small enough and doing a time rescaling, the simple zeros of $I(r) = I^+(r) - I^-(r)$, where

$$I^{\pm}(r) = \int_{\gamma_r^{\pm}} \frac{P_n^{\pm}(x,y)dy - Q_n^{\pm}(x,y)dx}{(x-1)^2 + y^2}, \tag{2}$$

gives limit cycles for (1), bifurcating from $\gamma_r^{\pm} = \{x^2 + y^2 = r^2 : \pm y > 0\}$. The above integrals defined over closed curves are known as Abelian integrals, see [4]. We can say that the expression (2) are the piecewise version of them. See more details in [10] or [19]. In our case both components of the unperturbed system have a common factor that appears in the denominator of the integrand. This expression appears in [18] (in polar coordinates) or in [11]. As we will see in Theorem 1.2 the explicit expression of (2) is different in the two period annuli associated to (1):

$$\mathcal{R}_i = \{r \in \mathbb{R} : 0 < r < 1\} \ \text{ and } \ \mathcal{R}_e = \{r \in \mathbb{R} : r > 1\}.$$

As we have commented before, the function $I(r)$ is also called the Abelian integral associated to system (1). By similarity we define the *inner and outer Abelian integrals* as