J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Boundedness of singular integrals on the flag Hardy spaces on Heisenberg group

Guorong Hu[a], Ji Li[b,*]

[a] Department of Mathematics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China
[b] Department of Mathematics, Macquarie University, NSW, 2109, Australia

A R T I C L E I N F O

Article history:
Received 16 February 2017
Available online xxxx
Submitted by M. Peloso

Keywords:
Discrete Littlewood–Paley analysis
Heisenberg group
Flag Hardy spaces
Singular integrals

A B S T R A C T

We prove that the classical one-parameter convolution singular integrals on the Heisenberg group are bounded on multiparameter flag Hardy spaces, which satisfy the 'intermediate' dilation between the one-parameter anisotropic dilation and the product dilation on $\mathbb{C}^n \times \mathbb{R}$ implicitly.

1. Introduction and statement of main results

The purpose of this note is to show that the classical one-parameter convolution singular integrals on the Heisenberg group are bounded on multiparameter flag Hardy spaces. Recall that the Heisenberg group $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R} = \{[z,t] : z \in \mathbb{C}^n, t \in \mathbb{R}\}$ and multiplication law

$$[z,t] \circ [z',t'] = [z_1, \cdots, z_n, t] \circ [z'_1, \cdots, z'_n, t'] := \Big[z_1 + z'_1, \cdots, z_n + z'_n, t + t' + 2\mathrm{Im}\big(\sum_{j=1}^{n} z_j \bar{z}_j\big)\Big].$$

The identity of $\mathbb{H}^n$ is the origin and the inverse is given by $[z,t]^{-1} = [-z,-t]$. Hereafter we agree to identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and to use the following notation to denote the points of $\mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$: $g = [z,t] \equiv [x,y,t] = [x_1, \cdots, x_n, y_1, \cdots, y_n, t]$ with $z = [z_1, \cdots, z_n]$, $z_j = x_j + iy_j$ and $x_j, y_j, t \in \mathbb{R}$ for $j = 1, \ldots, n$. Then, the composition law $\circ$ can be explicitly written as

$$g \circ g' = [x,y,t] \circ [x',y',t'] = [x + x', y + y', t + t' + 2\langle y, x'\rangle - 2\langle x, y'\rangle],$$

* Corresponding author.
E-mail addresses: hugr1984@163.com (G. Hu), ji.li@mq.edu.au (J. Li).

https://doi.org/10.1016/j.jmaa.2017.11.054
0022-247X/

Please cite this article in press as: G. Hu, J. Li, Boundedness of singular integrals on the flag Hardy spaces on Heisenberg group, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.11.054

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

Consider the dilations

$$\delta_r : \mathbb{H}^n \to \mathbb{H}^n, \quad \delta_r(g) = \delta_r([z,t]) = [rz, r^2 t].$$

A trivial computation shows that δ_r is an automorphism of $\mathbb{H}^n$ for every $r > 0$. Define a "norm" function ρ on $\mathbb{H}^n$ by

$$\rho(g) = \rho([z,t]) := \max\{|z|, |t|^{1/2}\}.$$

It is easy to see that $\rho(g^{-1}) = \rho(-g) = \rho(g)$, $\rho(\delta_r(g)) = r\rho(g)$, $\rho(g) = 0$ if and only if $g = 0$, and $\rho(g \circ g') \le \gamma(\rho(g) + \rho(g'))$, where $\gamma > 1$ is a constant.

The Haar measure on $\mathbb{H}^n$ is known to just coincide with the Lebesgue measure on $\mathbb{R}^{2n+1}$. For any measurable set $E \subset \mathbb{H}^n$, we denote by $|E|$ its (Haar) measure. The vector fields

$$T := \frac{\partial}{\partial t}, \quad X_j := \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t}, \quad Y_j := \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t}, \quad j = 1, \cdots, n,$$

form a natural basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$. For convenience we set $X_{n+j} := Y_j$ for $j = 1, 2, \cdots, n$, and set $X_{2n+1} := T$. Denote by $\widetilde{X}_j$, $j = 1, \cdots, 2n+1$, the right-invariant vector field which coincides with X_j at the origin. Let $\mathbb{N}$ be the set of all non-negative integers. For any multi-index $I = (i_1, \cdots, i_{2n+1}) \in \mathbb{N}^{2n+1}$, we set $X^I := X_1^{i_1} X_2^{i_2} \cdots X_{2n+1}^{i_{2n+1}}$ and $\widetilde{X}^I := \widetilde{X}_1^{i_1} \widetilde{X}_2^{i_2} \cdots \widetilde{X}_{2n+1}^{i_{2n+1}}$. It is well known that ([6])

$$X^I(f_1 * f_2) = f_1 * (X^I f_2), \quad \widetilde{X}^I(f_1 * f_2) = (\widetilde{X}^I f_1) * f_2, \quad (X^I f_1) * f_2 = f_1 * (\widetilde{X}^I f_2),$$

and

$$X^I \tilde{f} = (-1)^{|I|} \widetilde{\widetilde{X}^I f},$$

where $\tilde{f}$ is given by $\tilde{f}(g) := f(g^{-1})$. We further set

$$|I| := i_1 + \cdots + i_{2n+1} \quad \text{and} \quad d(I) := i_1 + \cdots + i_{2n} + 2i_{2n+1}.$$

Then $|I|$ is said to be the order of the differential operators X^I and $\widetilde{X}^I$, while $d(I)$ is said to be the homogeneous degree of X^I and $\widetilde{X}^I$.

Definition 1.1 *([14])*. A function ϕ is called a normalized bump function on $\mathbb{H}^n$ if ϕ is supported in the unit ball $\{g = [z,t] \in \mathbb{H}^n : \rho(g) \le 1\}$ and

$$|\partial_{z,t}^I \phi(z,t)| \le 1 \tag{1.1}$$

uniformly for all multi-indices $I \in \mathbb{N}^{2n+1}$ with $|I| \le N$, for some fixed positive integer N.

Remark 1.2. The condition (1.1) is equivalent (module a constant) to the following one:

$$|X^I \phi(g)| \le 1 \tag{1.2}$$

for all multi-indices I with $|I| \le N$. Indeed, this follows from the following the homogeneous property of the "norm" ρ and the fact that

Please cite this article in press as: G. Hu, J. Li, Boundedness of singular integrals on the flag Hardy spaces on Heisenberg group, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.11.054