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Cohen class of time-frequency representations and operators: Boundedness and uncertainty principles

Paolo Boggiatto, Evanthia Carypis, Alessandro Oliaro *

Department of Mathematics, University of Torino, Italy

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ABSTRACT

This paper presents a proof of an uncertainty principle of Donoho–Stark type involving ε-concentration of localization operators. More general operators associated with time-frequency representations in the Cohen class are then considered. For these operators, which include all usual quantizations, we prove a boundedness result in the L^p functional setting and a form of uncertainty principle analogous to that for localization operators.

1. Introduction

Uncertainty principles (UP) appear in harmonic analysis and signal theory in a variety of different forms involving not only the couple $(f, \widehat{f})$ formed by a signal (function or distribution) and its Fourier transform, but essentially every representation of a signal in the time-frequency space. Among the wide literature on this topic we refer for example to [2–4,8,11,14,15,17,19,21,22].

In this paper we consider the case where the couple $(f, \widehat{f})$ is substituted by a couple $(T_1 f, T_2 f)$, where T_1, T_2 are operators by which, in some sense, the concentration of the signal f is "tested". The consequent uncertainty statement is then of the following type: if the tests yield functions which are sufficiently concentrated on some domains of the time-frequency space, then the Lebesgue measure of these domains can not be "too small".

We make now precise the type of operators that are used, in which sense "concentration" is intended, and what is meant by "too small".

The class of operators that we consider is strictly connected with the *Cohen class* of time-frequency representations, which consists of sesquilinear forms of the type

$$Q_\sigma(f, g)(x, \omega) = \sigma * \mathrm{Wig}(f, g)(x, \omega), \tag{1}$$

* Corresponding author.
E-mail addresses: paolo.boggiatto@unito.it (P. Boggiatto), evanthia.carypis@unito.it (E. Carypis), alessandro.oliaro@unito.it (A. Oliaro).

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where

$$\mathrm{Wig}(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot t} f\,(x+t/2)\,\overline{g\,(x-t/2)}\,dt \tag{2}$$

is the *Wigner transform* and σ is the *Cohen kernel*. We shall shortly write $Q_\sigma(f)$ for the quadratic form $Q_\sigma(f,f)$. Clearly the signals f,g must be chosen in functional or distributional spaces such that the convolution (1) makes sense.

The Cohen class finds its justification in applied signal analysis as it actually coincides with the class of quadratic *covariant* time-frequency representations. More precisely, let Q be any sesquilinear form (non a priori in the Cohen class); a very natural requirement is that a translation in time $\tau_a f(x) = f(x-a)$ of the signal should reflect into the same translation of its representation along the time-axis, i.e. $Q(\tau_a f)(x,\omega) = Qf(x-a,\omega)$. On the other hand a modulation $\mu_b f(x) = e^{2\pi i b x} f(x)$ should reflect into a translation by the same parameter b along the frequency-axis, i.e. $Q(\mu_b f)(x,\omega) = Qf(x,\omega-b)$. It can be proved that these two requirements, called *covariance property*, actually characterize, under some minor technical hypothesis, the Cohen class among all quadratic representations (see e.g. [18], Thm. 4.5.1).

As described in [6], [7], we can associate an operator T^a_σ, depending on a symbol a, with each time-frequency representation Q_σ, by the formula:

$$(T^a_\sigma f, g) = (a, Q_\sigma(g,f)). \tag{3}$$

Formula (3) can be understood, e.g. in the Lebesgue setting, as follows:

$$Q_\sigma : L^q(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \to L^r(\mathbb{R}^{2d}),$$
$$T_\sigma : a \in L^{r'}(\mathbb{R}^{2d}) \to B\big(L^p(\mathbb{R}^d), L^{q'}(\mathbb{R}^d)\big),$$

where $1 < q < \infty,\ 1 < r \le \infty$, $1 \le p \le \infty$, and $\frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1$. For simplicity we write $\mathrm{Wig}\,(f)$ and $Q_\sigma(f)$ when $f = g$.

More generally, if $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, formula (3) defines a continuous linear map $T^a_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, and actually it establishes a bijection between operators and sesquilinear forms, we refer to [6] for details and general functional settings. The operators T^a_σ, obtained by (3) in correspondence with representations Q_σ in the Cohen class, will be called *Cohen operators*. Referring to (3), we actually remark that $(T^a_\sigma f, g) = (a, Q_\sigma(g,f)) = ((a * \overline{\tilde{\sigma}}, \mathrm{Wig}(g,f)))$, therefore, viewed as operators independently of quantization rules, all Cohen operators are Weyl operators (cfr. equation (9) and Proposition 14 (a)).

Due however to the freedom in the choice of the Cohen kernel σ, we recapture by (3) all types of quantizations used in pseudo-differential calculus (Weyl, Kohn–Nirenberg, localization, etc.). A particular family of operators of this kind is considered in [1], see Remark 13.

When the symbol a is the characteristic function of a measurable set in $\mathbb{R}^{2d}$ it is natural to look at Cohen operators as a generalized way of expressing the concentration of energy. In this spirit we shall consider couples of these operators applied to a signal f as the substitute for the couple $(f, \widehat{f})$ in the formulations of the UP of Donoho–Stark type in Sections 3 and 5. More precisely in Section 3 we shall consider the particular case of *localization operators*, see (8), correcting a flaw in the estimate of a Donoho–Stark type UP appearing in [5], whereas in Section 5 a similar UP in the general case of Cohen operators is presented. Sections 2 and 4 are dedicated to some L^p-boundedness results for Wigner (and Gabor) transforms and for general Cohen class operators respectively, which are preliminary to the results of the corresponding following sections.

Although a vast literature is available on L^p-boundedness, the norm estimates of Section 2 improve existing results as found in [9] and [29], and those in Section 4 furnish extensions of results for Weyl operators to Cohen operators.