J. Math. Anal. Appl. 461 (2018) 319–332

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a convexity problem in connection with some linear operators

Bogdan Gavrea

Department of Mathematics, Technical University of Cluj-Napoca, Str. Memorandumului nr. 28, 400114, Cluj-Napoca, Romania

ARTICLE INFO

Article history:
Received 14 January 2017
Available online 11 January 2018
Submitted by P. Nevai

Keywords:
Linear positive operators
Convex functions
Bernstein operators
Meyer-König and Zeller operators
Mirakyan–Favard–Szász operators

ABSTRACT

In this paper we give a generalization of the problem that was posed by I. Raşa in [10] and was proved based on a probabilistic approach by Mrowiec, Rajba and Wąsowicz in [8]. We end this paper by proposing two open problems related to Raşa's proposed inequality.

1. Introduction

Let $n \in \mathbb{N}$ and let Π_n denote the set of all polynomials of degree $\leq n$. The fundamental Bernstein polynomials of degree n are given by:

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ k = 0, 1, ..., n.$$

A long time ago, Professor I. Raşa ([10], Problem 2, p. 164), introduced the following problem: *Prove or disprove the following inequality:*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \left[b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y)\right] f\left(\frac{i+j}{2n}\right) \geq 0, \tag{1}$$

for any convex function $f \in C[0,1]$ and any $x, y \in [0,1]$. Using a probabilistic approach, J. Mirowiec, T. Rajba and S. Wąsowicz, [8], give a positive answer to the above problem and prove the following generalization of the inequality (1).

E-mail address: bogdan.gavrea@math.utcluj.ro.

https://doi.org/10.1016/j.jmaa.2018.01.010

Theorem 1.1 *([8], Theorem 12). Let $m, n \in \mathbb{N}$ with $m \geq 2$. Then,*

$$\sum_{i_1,\dots,i_m=0}^{n} [b_{n,i_1}(x_1)\dots b_{n,i_m}(x_1) + \dots + b_{n,i_1}(x_m)\dots b_{n,i_m}(x_m) \\ - m b_{n,i_1}(x_1)\dots b_{n,i_m}(x_m)]\, f\left(\frac{i_1 + \dots + i_m}{mn}\right) \geq 0, \tag{2}$$

for any convex function $f \in C[0,1]$ and any $x_1, \dots, x_m \in [0,1]$.

An elementary proof of (1), was given recently by Abel in [1]. In [1], the author shows that a (1) type inequality holds also for the Mirakyan–Favard–Szász ([1], Theorem 5) and for the Baskakov operators ([1], Theorem 6).

Following the ideas exposed in [1] by Abel, but using different methods, we plan to generalize the problem given by Raşa. More precisely, the aim of this paper is to obtain necessary and sufficient conditions for a wide class of operators, including Bernstein, Mirakyan–Favard–Szász and Meyer-König and Zeller type operators, for which (2) type inequalities hold.

Throughout this paper, we will use the following notations: x_+ denotes the nonnegative part of x,

$$x_+ = \frac{x + |x|}{2},$$

e_k is the k-th order monomial, $e_k(x) = x^k$ and $[t_0, .., t_m; f]$ denotes the divided difference of the function f on the distinct points $t_0, \dots, t_m$.

2. Generating functions and the corresponding linear positive operators

Let I be one of the intervals $[0, \infty)$ or $[0, 1]$. Let $g_n : I \times D \to \mathbb{C}$, $D = \{z \in \mathbb{C} \mid |z| \leq R\}$, $R > 1$ be a function with the property that for any fixed $x \in I$, the function $g_n(x, \cdot)$ is an analytic function on D,

$$g_n(x, z) = \sum_{k=0}^{\infty} a_{n,k}(x) z^k \tag{3}$$

and $a_{n,k}(x) \geq 0$ for all $k \geq 0$ and for all $x \in I$.

Let $\mathcal{F}$ be a linear set of functions defined on the interval I and let $\{A_t\}_{t\in[0,\infty)}$ be a set of real linear positive functionals defined on $\mathcal{F}$ with the property that for any $f \in \mathcal{F}$, the series

$$L_n(f)(x) := \sum_{k=0}^{\infty} a_{n,k}(x) A_{\frac{k}{n}}(f), \tag{4}$$

is convergent for any $x \in I$. The identity (4) defines a linear positive operator. The function g_n will be referred to as the *generating function* for the operators L_n, relative to the set of functionals $\{A_t\}_{t\in I}$.

3. Main results. The two-dimensional case

The aim of this section is to establish more general inequalities of type (1). The main result is given by Theorem 3.1. In order to prove this theorem two lemmas Lemma 3.1 and Lemma 3.2 are used. Several other useful results are given in Corollary 3.1 and Corollary 3.2.