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A note on Anderson's theorem in the infinite-dimensional setting

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ABSTRACT

Anderson's theorem states that if the numerical range $W(A)$ of an n-by-n matrix A is contained in the unit disk $\overline{\mathbb{D}}$ and intersects with the unit circle at more than n points, then $W(A) = \overline{\mathbb{D}}$. An analogue of this result for compact A in an infinite dimensional setting was established by Gau and Wu. We consider here the case of A being the sum of a normal and compact operator.

1. Introduction

The *numerical range* (also known as the *field of values*, or the *Hausdorff set*) of a bounded linear operator A acting on a Hilbert space $\mathcal{H}$ is defined as

$$W(A) = \{\langle Ax, x\rangle \colon \|x\| = 1\}.$$

Here $\langle .,.\rangle$ and $\|.\|$ stand for the scalar product on $\mathcal{H}$ and the norm generated by it, respectively.

The set $W(A)$ is a convex (Toeplitz–Hausdorff theorem), bounded, and in the case $\dim \mathcal{H} < \infty$ also closed subset of the complex plane $\mathbb{C}$.

We will use the standard notation $\overline{X}, X^o, \partial X, X'$ for the closure, interior, the boundary, and the set of the limit points, respectively, of subsets $X \subset \mathbb{C}$. In particular, $\mathbb{D} = \{z \colon |z| < 1\}$ is the open unit disk, $\partial\mathbb{D} = \mathbb{T}$ is the unit circle, and $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$ is the closed unit disk.

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The closure $\overline{W(A)}$ of the numerical range of A contains the spectrum $\sigma(A)$, and thus the convex hull $\operatorname{conv}\sigma(A)$ of the latter. For normal A, $\overline{W(A)} = \operatorname{conv}\sigma(A)$. We refer to [4] for these and other well known properties of the numerical range.

Anderson's theorem (unpublished by Joel Anderson himself but discussed e.g. in [2,7]) states that if $W(A)$ is contained in $\overline{\mathbb{D}}$ and the intersection of $W(A)$ with $\mathbb{T}$ consists of more than $n = \dim \mathcal{H}$ points, then in fact $W(A) = \overline{\mathbb{D}}$. This result is sharp in a sense that for a unitary operator U with a simple spectrum acting on an n-dimensional $\mathcal{H}$, $W(U)$ is a polygon with n vertices on $\mathbb{T}$ and thus different from $\overline{\mathbb{D}}$.

Unitary operators also deliver easy examples showing that Anderson's theorem does not generalize to the infinite-dimensional setting. Indeed, if A is a diagonal operator with the point spectrum $\sigma_p(U) = \{\lambda_j,\ j = 1, 2, \ldots\} \subset \mathbb{T}$, then $\overline{W(A)} = \operatorname{conv}\sigma_p(A) \subsetneq \overline{\mathbb{D}}$ while $W(A) \cap \mathbb{T} = \sigma_p(A)$ is infinite.

Moreover, according to [7] every bounded convex set G for which $G \setminus G^o$ is the union of countably many singletons and conic arcs is the numerical range of some operator acting on a separable $\mathcal{H}$.

On the positive side, Anderson's theorem generalizes quite naturally to the infinite dimensional case under some restrictions on the operators involved. As was shown more recently in [3], the following result holds:

Theorem 1. *If A is a compact operator on a Hilbert space with $W(A)$ contained in $\overline{\mathbb{D}}$ and $\overline{W(A)}$ intersecting $\mathbb{T}$ at infinitely many points, then $W(A) = \overline{\mathbb{D}}$.*

In this paper, we single out a wider class of operators for which analogs of Anderson's theorem are valid in an infinite dimensional setting.

2. Main results

We start with a lemma.

Lemma 2. *Let $A = N + K$, where N is normal and K is a compact operator on a Hilbert space $\mathcal{H}$. If $W(A) \subset \overline{\mathbb{D}}$ and γ is a closed arc of $\mathbb{T}$ such that the intersection $\gamma \cap \overline{W(A)}$ is infinite while $\gamma \cap \sigma_{ess}(A) = \emptyset$, then $\gamma \subset W(A)$.*

Recall that the essential spectrum $\sigma_{ess}(A)$ of an operator A is the set of $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not Fredholm. Equivalently, $\sigma_{ess}(A)$ is the spectrum of the equivalence class of A in the Calkin algebra of the algebra of bounded linear operators by the ideal of compact operators.

The proof of this lemma is delegated to the next section; we will discuss here some of its consequences.

Theorem 3. *Let $A = N + K$, where N is normal and K is a compact operator on a Hilbert space $\mathcal{H}$. Let also $W(A) \subset \overline{\mathbb{D}}$ and Γ be a (relatively) open subset of $\mathbb{T}$ disjoint with $\sigma_{ess}(A)$. If every connected component of Γ contains limit points of its intersection with $\overline{W(A)}$, then $\Gamma \subset W(A)$.*

Proof. Connected components of Γ are open arcs Γ_j. Writing Γ_j as $\bigcup_{k=1}^{\infty} \gamma_{jk}$, where

$$\gamma_{j1} \subset \gamma_{j2} \subset \cdots \subset \gamma_{jk} \subset \cdots$$

is an expanding family of closed arcs, we see that $\gamma = \gamma_{jk}$ satisfy the conditions of Lemma 2 and thus $\gamma_{jk} \subset W(A)$, for k large enough. Consequently,

$$\Gamma = \cup_{j,k=1}^{\infty} \gamma_{jk} \subset W(A). \quad \square$$