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A characterization of minimal Orlicz–Sobolev norms in the affine class ☆

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ABSTRACT

The Orlicz–Sobolev norm is invariant only under rigid motions but is not invariant under volume preserving affine transformations on $\mathbb{R}^n$. In this paper, we will consider the Orlicz–Sobolev norm in the affine class and characterize its minimum.

1. Introduction

Let $\phi : [0, \infty) \to [0, \infty)$ be a convex, strictly increasing, continuously differentiable function on $(0, \infty)$ with $\phi(0) = 0$. The Orlicz space $L^\phi(\mathbb{R}^n)$ associated with the function ϕ is the space equipped with the Luxemburg norm defined as

$$\|f\|_\phi = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\Big\}, \tag{1.1}$$

for any measurable function f on $\mathbb{R}^n$. If $|f|$ is replace by the length of the gradient ∇f, then the Orlicz–Sobolev norm can be defined by

$$\|\nabla f\|_\phi = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\Big(\frac{|\nabla f(x)|}{\lambda}\Big) dx \le 1\Big\}. \tag{1.2}$$

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The Orlicz–Sobolev space equipped with the Orlicz–Sobolev norm extends the usual Sobolev space in that the role of Lebesgue space is played by the more general Orlicz space. In other words, the Orlicz–Sobolev space $W^{1,\phi}(\mathbb{R}^n)$ is the set of all weakly differentiable functions in $L^\phi(\mathbb{R}^n)$ whose gradient belongs to $L^\phi(\mathbb{R}^n)$. Nowadays, this space has been investigated extensively, see e.g., [1–17,19–22,25–29].

Notice that the Orlicz–Sobolev norm is invariant only under rigid motions but is not invariant under volume preserving affine transformations on $\mathbb{R}^n$. In this paper, we will consider the Orlicz–Sobolev norm in the affine class and characterize its minimum. Denote $f_T(x) = f(Tx)$ for every $T \in \mathrm{SL}(n)$ and $x \in \mathbb{R}^n$. The existence of the minimal Orlicz–Sobolev norm in the affine class is as follows.

Theorem 1.1. *Suppose $f \in W^{1,\phi}(\mathbb{R}^n)$ is not equal to a constant function almost everywhere. Then there exists a unique (up to orthogonal transformations) $T_\phi \in \mathrm{SL}(n)$ which minimizes $\{\|\nabla f_T\|_\phi : T \in \mathrm{SL}(n)\}$.*

The minimal affine Orlicz–Sobolev norm is characterized in the following.

Theorem 1.2. *Suppose $f \in W^{1,\phi}(\mathbb{R}^n)$ is not equal to a constant function almost everywhere and the integral $\int_{\mathbb{R}^n} |\nabla f(y)|\phi'\Big(\frac{|\nabla f(y)|}{\|\nabla f\|_\phi}\Big)dy$ exists. Then*

$$\|\nabla f\|_\phi = \min_{T\in \mathrm{SL}(n)} \|\nabla f_T\|_\phi \tag{1.3}$$

if and only if

$$\int_{\mathbb{R}^n} \nabla f(x) \otimes \nabla f(x) \frac{1}{|\nabla f(x)|}\phi'\Big(\frac{|\nabla f(x)|}{\|\nabla f\|_\phi}\Big)dx = \frac{1}{n}\int_{\mathbb{R}^n} |\nabla f(x)|\phi'\Big(\frac{|\nabla f(x)|}{\|\nabla f\|_\phi}\Big)dx I_n, \tag{1.4}$$

where $\nabla f \otimes \nabla f$ is the rank-one orthogonal projection onto the space spanned by the vector ∇f and I_n denotes the identity operator on $\mathbb{R}^n$. Moreover, modulo orthogonal transformations, I_n is the unique operator which minimizes $\{\|\nabla f_T\|_\phi : T \in \mathrm{SL}(n)\}$.

Note that, if $\phi(t) = t^p$ for $p \geq 1$, then $L^\phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $W^{1,\phi}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$. The Orlicz–Sobolev norm $\|\nabla f\|_\phi$ reduces to the usual L_p-Sobolev norm $\|\nabla f\|_p$. Notice that the integral $\int_{\mathbb{R}^n} |\nabla f(y)|\phi'\Big(\frac{|\nabla f(y)|}{\|\nabla f\|_\phi}\Big)dy = p\|\nabla f\|_p^{p-1}$, and we can remove its existence in Theorem 1.2. Thus, we immediately obtain the following corollary due to the authors [18] (the case $p = 2$) and Lutwak, Lv, Yang, Zhang [24]. The 'only if' part is new.

Corollary 1.3. *Suppose $p \geq 1$ and $f \in W^{1,p}(\mathbb{R}^n)$ is not equal to a constant function almost everywhere. Then*

$$\|\nabla f\|_p = \min_{T\in \mathrm{SL}(n)} \|\nabla f_T\|_p,$$

if and only if

$$\int_{\mathbb{R}^n} |\nabla f(x)|^{p-2}\nabla f(x) \otimes \nabla f(x)dx = \frac{1}{n}\int_{\mathbb{R}^n} |\nabla f(x)|^p dx I_n. \tag{1.5}$$

Moreover, modulo orthogonal transformations, I_n is the unique operator which minimizes $\{\|\nabla f_T\|_p : T \in \mathrm{SL}(n)\}$.

Besides, we shall mention that the proof in [18, Theorem 4.3] can also be used to obtain identity (1.5) by taking $\Phi(v)^p = |u \cdot v|^2|v|^{p-2}$.

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