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# Inverse problem with finite overdetermination for steady-state equations of radiative heat exchange

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An inverse problem for a system of semilinear elliptic equations modeling simultaneously conductive and radiative heat transfer is under consideration. The problem consists in finding the right-hand side of the heat transfer equation, in the form of linear combination of given functionals, on the base of prescribed values of these functionals on the solution. The solvability of the problem is proven without any smallness assumptions. It is shown that the set of solutions is homeomorphic to a finite-dimensional compact set, and conditions of uniqueness of solutions are found.


## 1. Introduction

The interest in studying problems of complex heat transfer [15], where the radiative and conductive contributions are simultaneously taken into account, is motivated by their importance for many applications (e.g., glass manufacturing [7], laser-interstitial thermotherapy [18], design of cooling systems [17], etc.).

The radiative heat transfer equation (RHTE) is a first order integro-differential equation governing the radiation intensity. A way of simplification of the integro-differential RHTE is the representation of the local intensity by series of spherical harmonics, truncated after $N$ terms, and substitution of such an ansatz into the moments of the differential form of the RHTE (see e.g. [15]). This approach leads to the so-called $P_N$ approximation, where $N$ is the number of terms used in the expansion. Especially interesting is the $P_1$, diffusion, approximation which does not require high computational efforts.

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The $P_1$ approximation of homogeneous steady-state radiative–conductive heat transfer models is relatively well investigated. In [11,10,13], the diverse boundary-value problems are studied. Papers [16,6,14,12,9] are concerned with inverse extremal problems of complex heat transfer.

It should be noted that the analysis of equations of complex heat transfer in the presence of inner heat and radiation sources requires obtaining new a priori estimates of solutions. In [8], the above mentioned estimates are obtained for transient equations of complex heat transfer. In [17], an inhomogeneous boundary-value problem for the $SP_3$ approximation of an RHTE-based heat transfer model is studied. In the case of bounded temperature sources, the unique solvability of the problem is proven.

In the current paper, an inverse problem for a radiative–conductive heat transfer model based on the $P_1$ approximation is under consideration. In contrast to [17], we introduce right-hand side terms into the heat transfer equation, not assuming that these terms are necessary modeling bounded volume or surface heat sources. The problem consists in finding the right-hand side sources of the heat transfer equation in the form of linear combination of given functionals using prescribed values of these functionals on unknown solutions (finite-dimensional overdetermination). The solvability of the problem is proven without any smallness assumptions. It is shown that the set of solutions is homeomorphic to a finite-dimensional compact set, and conditions of uniqueness of solutions are found.

Notice that inverse problems with finite-dimensional overdetermination have been considered for Navier–Stokes equations, see [1]. In paper [2], an inverse problem of finding surface heat fluxes from mean temperatures governed by free-convection equations for viscose incompressible fluid. In [3–5], inverse problems with finite overdetermination are studied in concern with different models of continuum media.

## 2. Problem formulation

The steady-state process of complex heat transfer in a bounded domain $\Omega \subset \mathbb{R}^3$ is modeled by the following system of semilinear elliptic equations, see [15,13]:

$$-a\Delta\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = f,\ -\alpha\Delta\varphi + \kappa_a(\varphi - |\theta|\theta^3) = 0. \tag{1}$$

Here, $\theta$ is the normalized temperature, and $\varphi$ is the normalized radiation intensity averaged over all directions. Positive physical parameters $a$, $b$, $\kappa_a$, and $\alpha$, describing properties of the medium, are defined in a conventional way, see [13]. The right-hand side $f$ of the heat transfer equation describes volume or surface heat sources in $\Omega$. The system of equations (1) is supplied by the following boundary conditions on $\Gamma = \partial\Omega$:

$$a\partial_n\theta + \beta(\theta - \theta_b) = 0, \qquad \alpha\partial_n\varphi + \gamma(\varphi - \theta_b^4) = 0. \tag{2}$$

Here, $\partial_n$ denotes the normal derivative taken with respect to the exterior normal $\mathbf{n}$ on $\Gamma$. Nonnegative functions $\theta_b$, $\beta$, and $\gamma$ are given and fixed.

To formulate the inverse problem, write the boundary value problem (1) and (2) in an operator form. In the following, it is assumed that $\Omega$ is a Lipschitzian bounded domain. Let $L^p$, $1 \le p \le \infty$, denote the corresponding Lebesgue space, whereas $H^s$ stands for the Sobolev space $W_2^s$. Denote $H = L^2(\Omega)$ and $V = H^1(\Omega)$. Let $V'$ be the dual space of $V$. As usually, the space $H$ is identified with $H'$ so that $V \subset H = H' \subset V'$. Denote, respectively, by $\|\cdot\|$, $\|\cdot\|_V$, and $\|\cdot\|_{V'}$ the norms in $H$, $V$, and $V'$. For any $g \in V'$ and $v \in V$, let $(g, v)$ be the value of the functional $g \in V'$ on the element $v \in V$. Note that $(g, v)$ coincides with the inner product of $H$ if $g \in H$.

Assume that the following conditions hold:

$(i)$ $\beta, \gamma \in L^\infty(\Gamma)$, $\beta \ge \beta_0 > 0$, $\gamma \ge \gamma_0 > 0$, $\beta_0, \gamma_0 = \text{Const}$, $\theta_b \in L^\infty(\Gamma)$, $f \in V'$.