J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Flag representations of mixed volumes and mixed functionals of convex bodies ☆

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ARTICLE INFO

Article history:
Received 15 May 2017
Available online xxxx
Submitted by J. Bastero

Keywords:
Mixed volumes
Mixed functionals
Curvature measures
Flag measures
Grassmannian
Integral geometry

ABSTRACT

Mixed volumes $V(K_1,\dots,K_d)$ of convex bodies $K_1,\dots,K_d$ in Euclidean space $\mathbb{R}^d$ are of central importance in the Brunn–Minkowski theory. Representations for mixed volumes are available in special cases, for example as integrals over the unit sphere with respect to mixed area measures. More generally, in Hug–Rataj–Weil (2013) [11] a formula for $V(K[n],M[d-n])$, $n\in\{1,\dots,d-1\}$, as a double integral over flag manifolds was established which involved certain flag measures of the convex bodies K and M (and required a general position of the bodies). In the following, we discuss the general case $V(K_1[n_1],\dots,K_k[n_k])$, $n_1+\dots+n_k=d$, and show a corresponding result involving the flag measures $\Omega_{n_1}(K_1;\cdot),\dots,\Omega_{n_k}(K_k;\cdot)$. For this purpose, we first establish a curvature representation of mixed volumes over the normal bundles of the bodies involved. We also obtain a corresponding flag representation for the mixed functionals from translative integral geometry and a local version, for mixed (translative) curvature measures.

1. Introduction

Mixed volumes of convex bodies build a basic concept and tool in the Brunn–Minkowski theory of convex geometry. They arise by combining two fundamental geometric notions, the Minkowski addition of sets and the volume functional V_d. Namely, for convex bodies $K_1,\dots,K_k$ (non-empty compact convex sets) in $\mathbb{R}^d$,

☆ The first and the third author are grateful for support from the German Science Foundation (DFG), grant No. HU 1874/2-2, HU 1874/3-2, HU 1874/4-2, WE 1613/2-2, FOR 1548. The second author was supported by the Czech Science Foundation, project No. P201/15-08218S.

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https://doi.org/10.1016/j.jmaa.2017.12.039

Please cite this article in press as: D. Hug et al., Flag representations of mixed volumes and mixed functionals of convex bodies, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.12.039

$d \geq 2$, and numbers $t_1, \dots, t_k \geq 0$, the volume of the linear combination $t_1K_1 + \cdots + t_kK_k$ (which is again a convex body) is a (homogeneous) polynomial in $t_1, \dots, t_k$, that is

$$V_d(t_1K_1 + \cdots + t_kK_k) = \sum_{i_1=1}^{k} \cdots \sum_{i_d=1}^{k} t_{i_1} \cdots t_{i_d} V(K_{i_1}, \dots, K_{i_d}). \tag{1}$$

The coefficients $V(K_{i_1}, \dots, K_{i_d})$ are assumed to be symmetric and are therefore uniquely determined. Moreover, $V(K_{i_1}, \dots, K_{i_d})$ is linear in each of its entries $K_{i_1}, \dots, K_{i_d}$. For further basic properties of mixed volumes and all other notions from convex geometry which we use, we refer to the book [19]. As usual, we abbreviate by $V(K_1[n_1], \dots, K_k[n_k])$ the mixed volume where the body K_i appears n_i times, for $i = 1, \dots, k$, and $n_1 + \cdots + n_k = d$. The functional $V(K_1[n_1], \dots, K_k[n_k])$ is homogeneous of degree n_i in K_i.

In [11], it was shown that

$$V(K[n], M[d-n]) = \iint f_{n,d-n}(u, U, v, V)\, \Omega_n(K; d(u, U))\, \Omega_{d-n}(M; d(v, V)), \tag{2}$$

where $\Omega_n(K; \cdot)$ and $\Omega_{d-n}(M; \cdot)$ are flag measures of K and M, respectively, the function $f_{n,d-n}$ is independent of K and M, and the integration is over the manifold of flags (u, U) (respectively (v, V)). For this formula, we had to assume that K and M are in general relative position with respect to each other. If K and M are polytopes, this condition is, for instance, satisfied if K and M do not have parallel faces of complementary dimension. The proof of (2) was based on a curvature representation of mixed functionals from translative integral geometry which was proved in [17] and used the fact that the mixed volume $V(K[n], -M[d-n])$ and the mixed functional $V_{n,d-n}(K, M)$ from translative integral geometry coincide (up to a binomial coefficient).

The iteration of translative integral formulas yields an expansion which resembles (1) but involves mixed functionals of a different nature. Namely,

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} V_j(K_1 \cap (K_2 + z_2) \cap \cdots \cap (K_k + z_k))\, \mathcal{H}^d(dz_k) \cdots \mathcal{H}^d(dz_2)$$

$$= \sum_{\substack{r_1, \dots, r_k = j \\ r_1 + \cdots + r_k = (k-1)d + j}}^{d} V_{r_1, \dots, r_k}(K_1, \dots, K_k) \tag{3}$$

for $j = 0, \dots, d$, where $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure. Translative integral formulas are at the basis of integral geometry and have important applications in stochastic geometry. We refer to [20, Section 6.4 and Chapter 9], for background information, and for details of such applications and for further references. Since j is determined by $j = r_1 + \cdots + r_k - (k-1)d$, we skipped the upper index (j) which was used in [20] and previous papers for the mixed functionals on the right-hand side of (3). We remark that $V_{r_1, \dots, r_k}(K_1, \dots, K_k)$ is symmetric in the bodies involved, as long as $K_1, \dots, K_k$ and $r_1, \dots, r_k$ undergo the same permutation. Moreover, if $r_i = 0$ (hence $j = 0$), then the mixed functional $V_{r_1, \dots, r_k}(K_1, \dots, K_k)$ does not depend on K_i, and if $r_k = d$, then

$$V_{r_1, \dots, r_k}(K_1, \dots, K_k) = V_{r_1, \dots, r_{k-1}}(K_1, \dots, K_{k-1}) V_d(K_k).$$

Hence, we may concentrate on the cases where $1 \leq r_1, \dots, r_k \leq d - 1$. Since $V_{r_1, \dots, r_k}(K_1, \dots, K_k)$ is homogeneous of degree r_i in K_i, $i = 1, \dots k$, the total degree of the mixed functional is $r_1 + \cdots + r_k = (k-1)d + j$. Therefore, for $k > 2$ or for $k = 2$ and $j > 0$, the mixed volume $V(K_1[n_1], \dots, K_k[n_k])$ and the mixed functional $V_{r_1, \dots, r_k}(K_1, \dots, K_k)$ have completely different homogeneity properties. In fact, apart

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