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Asymptotics for polynomials orthogonal in an indefinite metric

Maxim Derevyagin [a,*], Brian Simanek [b]

[a] *University of Mississippi, Department of Mathematics, Hume Hall 305, P. O. Box 1848, University, MS 38677-1848, USA*
[b] *Baylor University, Department of Mathematics, One Bear Place #97328, Waco, TX 76798-7328, USA*

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A B S T R A C T

We continue studying polynomials generated by the Szegő recursion when a finite number of Verblunsky coefficients lie outside the closed unit disk. We prove some asymptotic results for the corresponding orthogonal polynomials and then translate them to the real line to obtain the Szegő asymptotics for the resulting polynomials. The latter polynomials give rise to a non-symmetric tridiagonal matrix but it is a finite-rank perturbation of a symmetric Jacobi matrix.

1. Introduction

In the theory of orthogonal polynomials, two distinguished classes have historically received special attention, namely that in which the measure of orthogonality is supported on the unit circle (OPUC) and that in which the measure of orthogonality is supported on the real line (OPRL). The prominent distinguishing feature of these classes is the existence of a recursion relation satisfied by the orthogonal polynomials. In the setting of OPUC, this gives rise to the sequence of so-called *Verblunsky coefficients*, which we denote by $\{\alpha_n\}_{n=0}^{\infty}$, each of which is a complex number in the open unit disk. Verblunsky's Theorem establishes a bijection between such sequences and infinitely supported probability measures on the unit circle (see [16, Chapter 1]). Similarly, Favard's Theorem establishes a bijection between pairs of bounded real sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ where each $a_n > 0$ and probability measures with infinite and compact support on the real line (see [18, Theorem 1.3.7]). A common theme of the research in these fields has been to investigate the relationship between the measure of orthogonality and the corresponding

* Corresponding author.
E-mail addresses: derevyagin.m@gmail.com (M. Derevyagin), Brian_Simanek@baylor.edu (B. Simanek).

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sequence or sequences (see [16–18]). We will refer to any sequence or sequences to which we can apply Verblunsky's Theorem or Favard's Theorem as belonging to the *classical case*.

In [7], a special non-classical class of Verblunsky coefficients was studied and unlike the classical case, that class does not correspond to measures on the unit circle $\mathbb{T}$. In this note we proceed with the exploration of this class and we will use the notation from [7], which is in turn inherited from [16,17]. More precisely, we consider sequences $\{\alpha_n\}_{n=0}^{\infty}$ of complex numbers for which there exists a natural number N such that

$$\begin{aligned} &|\alpha_n| \neq 1, \quad n = 0, 1, 2, \dots, N-1, \\ &|\alpha_n| < 1, \quad n = N, N+1, N+2, \dots. \end{aligned} \tag{1.1}$$

Due to Verblunsky's theorem, a sequence $\{\alpha_n\}_{n=0}^{\infty}$ that satisfies (1.1) does not correspond to a positive measure on $\mathbb{T}$ if even a single α_n lies outside the unit disk. Consequently, we cannot associate the L^2-space of a positive measure on $\mathbb{T}$ to such a sequence. However, we will see that one can still associate a function space equipped with a sesquilinear form to such a sequence and, therefore, develop a notion of orthogonality in this space. The existing literature sometimes refers to this space as a space with an indefinite metric and one can develop a theory of polynomials that are orthogonal in this indefinite metric (for instance, see [14], where the author considers a situation similar to ours).

If $\{\alpha_n\}_{n=0}^{\infty}$ satisfies (1.1), then for any nonnegative integer n it is still possible to define a monic polynomial Φ_{n+1} of degree $n+1$ by iterating the Szegő recurrence

$$\begin{aligned} \Phi_{n+1}(z) &= z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z) \\ \Phi_{n+1}^*(z) &= \Phi_n^*(z) - \alpha_n z \Phi_n(z), \end{aligned} \tag{1.2}$$

provided that we set the initial condition to be

$$\Phi_0(z) = 1 \tag{1.3}$$

and Φ_n^* is the polynomial reversed to Φ_n, that is,

$$\Phi_n^*(z) = z^n \overline{\Phi_n(1/\overline{z})}. \tag{1.4}$$

Given a sequence $\{\alpha_n\}_{n=0}^{\infty}$, we will often refer to the m-times stripped sequence given by $\{\alpha_n\}_{n=m}^{\infty}$. If a sequence satisfies (1.1), then the N-times stripped sequence satisfies the hypotheses of Verblunsky's theorem and thus corresponds to a measure on the unit circle. Associated to such a measure is the sequence $\{f_n\}_{n=N}^{\infty}$ of Schur functions that satisfy the recursive relation

$$f_n(z) = \frac{\alpha_n + z f_{n+1}(z)}{1 + \bar{\alpha}_n z f_{n+1}(z)}, \quad n = N, N+1, N+2, \dots. \tag{1.5}$$

This recursion can be iterated with any choice of $\{\alpha_n\}_{n=0}^{N-1}$ and hence to any sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (1.1) we can associate a sequence of functions $\{f_n\}_{n=0}^{\infty}$ that obeys (1.5). Since we allow some Verblunsky coefficients to be outside the closed unit disk, in this situation one cannot say that all f_n's are Schur functions. However, (1.1) ensures that f_n is a Schur function for $n = N, N+1, \dots$. Once we have the sequence of functions $\{f_n\}_{n=0}^{\infty}$, it is natural to define a function that will play the role of a Carathéodory function in our theory. To this end, let us set $f = f_0$ and use the standard formula to define F as in [7]

$$F(z) := \frac{1 + z f(z)}{1 - z f(z)}. \tag{1.6}$$

As is shown in [7, Proposition 2.1], Khrushchev's formula still holds for this F, that is,

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