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Approximate symmetry of Birkhoff orthogonality

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ABSTRACT

In a real normed space X we consider an approximate symmetry of the Birkhoff orthogonality $\perp_B$ and establish its connections with some properties of the space X. Moreover, we introduce and study a new geometric constant for X, connected with the considered property.

0. Introduction

The *Birkhoff orthogonality* in a normed space is generally not a symmetric relation. However, allowing some inaccuracy one can consider an *approximate Birkhoff orthogonality* (a new characterization of it has been obtained recently in [8]) and then a related notion of *approximate symmetry* of this orthogonality can be introduced and investigated. In the present paper we deal with this problem. We consider also some geometrical properties connected with such an approximate symmetry.

Throughout the paper we consider a real normed space $(X, \|\cdot\|)$ with $\dim X \geq 2$. B_X and S_X stand for the closed unit ball and the unit sphere in X, respectively. For another normed space Y, $\mathcal{L}(X,Y)$ denotes the space of all linear and bounded operators from X into Y. For $T \in \mathcal{L}(X,Y)$, by M_T we mean the set of unit vectors at which T attains its norm, namely $M_T := \{x \in S_X : \ \|Tx\| = \|T\|\}$. $\mathcal{K}(X,Y)$ is the subspace of $\mathcal{L}(X,Y)$ consisting of all compact operators. X^* stands for the dual and X^{**} for the bidual space of X.

1. Approximate Birkhoff orthogonality

The *Birkhoff orthogonality* in X is defined by:

$$x \perp_B y \quad \iff \quad \|x + \lambda y\| \geq \|x\|, \ \forall\, \lambda \in \mathbb{R}$$

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(cf. [3,14,15] or a survey [1]). We consider also an *approximate Birkhoff orthogonality* (ε*-Birkhoff orthogonality* with $\varepsilon \in [0,1)$):

$$x \perp_B^\varepsilon y \quad \iff \quad \|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\,\|\lambda y\|, \ \forall\,\lambda \in \mathbb{R} \tag{1.1}$$

as introduced in [5]. Obviously, $\perp_B^0 = \perp_B$. If X is an inner product space, the approximate orthogonality is naturally defined by:

$$x \perp^\varepsilon y \quad \iff \quad |\,\langle x|y\rangle\,| \leq \varepsilon\,\|x\|\,\|y\|,$$

and in this case $\perp_B^\varepsilon$ coincides with $\perp^\varepsilon$ (cf. [5, Proposition 2.1]).

In a recent paper [8] the authors have proved the following characterization of the approximate Birkhoff orthogonality.

Theorem 1.1 *([8], Theorem 2.3). Let X be a real normed space. For $x, y \in X$ and $\varepsilon \in [0,1)$:*

$$x \perp_B^\varepsilon y \quad \iff \quad \exists z \in \mathrm{Lin}\{x,y\} \ s.t. \ x \perp_B z, \ \|z-y\| \leq \varepsilon\|y\|. \tag{1.2}$$

Another characterization can be derived from Theorem 1.1 by using the supporting functionals at $x \in X$:

$$J(x) := \{\varphi \in X^* : \quad \|\varphi\| = 1, \ \varphi(x) = \|x\|\}.$$

Theorem 1.2 *([8], Theorem 2.4). Let X be a real normed space. For $x, y \in X$ and $\varepsilon \in [0,1)$:*

$$x \perp_B^\varepsilon y \quad \iff \quad \exists\,\varphi \in J(x) \ s.t. \ |\varphi(y)| \leq \varepsilon\|y\|. \tag{1.3}$$

Clearly, (1.3) generalizes James' characterization (cf. [15, Corollary 2.2]):

$$x \perp_B y \quad \iff \quad \exists \varphi \in J(x) \text{ s.t. } \varphi(y) = 0. \tag{1.4}$$

A different definition of an approximate Birkhoff orthogonality was given by Dragomir [9]. For a given $\varepsilon \in [0,1)$ and $x, y \in X$:

$$x \underset{\varepsilon}{\perp}_B y \quad \iff \quad \|x+\lambda y\| \geq (1-\varepsilon)\|x\|, \ \forall\,\lambda \in \mathbb{R}. \tag{1.5}$$

Some relationships between definitions (1.1) and (1.5) were established in [5] and [19]. In particular, for inner product spaces $\underset{\varepsilon}{\perp}_B$ is equal to $\perp^\eta$ with $\eta = \sqrt{1-(1-\varepsilon)^2}$.

In an arbitrary normed space Proposition 3.1 in [19] yields

$$x \perp_B^\varepsilon y \quad \Rightarrow \quad x \underset{\delta}{\perp}_B y,$$

with $\delta := 1 - \sqrt{1-4\varepsilon}$ and for $\varepsilon \leq \frac{1}{4}$. We will improve this result for real spaces.

Theorem 1.3. *Let X be a real normed space and let $\varepsilon \in \left[0, \frac{1}{2}\right)$. Then*

$$x \perp_B^\varepsilon y \quad \Rightarrow \quad x \underset{2\varepsilon}{\perp}_B y. \tag{1.6}$$