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Symmetry and monotonicity of positive solution of elliptic equation with mixed boundary condition in a spherical cone

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ABSTRACT

In this paper we prove some symmetry results for positive solutions of the semilinear elliptic equations of the type $\Delta u + f(u) = 0$ with mixed boundary conditions in a spherical cone. In particular, we show that these solutions have a unique peak which is on the boundary of the spherical cone.

1. Introduction

In this paper we are concerned about the symmetric and the monotonicity of positive solution of the semilinear elliptic equation

$$d\Delta u + f(u) = 0 \tag{1.1}$$

with mixed boundary conditions in a spherical cone.

Problem (1.1) serves as a model in many different areas of applied mathematics and has been extensively investigated in the last three decades. For example, it appears in astrophysics as the nonlinear scalar field equation; in chemistry and population dynamics governing the stationary states; in fluid mechanics describing the blowup set of some porous-medium equations; and also in some models in plasma physics.

Symmetry and monotonicity are important topics in the modern theory of partial differential equations. There is a large literature on this topic and probably the most famous strategy used to study this question is the method of moving plane. The method of moving plane is introduced by A. D. Alexandrov [1] and Serrin [25] in the early 1970s, and developed by Gidas, Ni and Nirenberg [12] to obtain a celebrated symmetry result in bounded domains. In their well-known paper [12], they show that all positive solutions of the Dirichlet problem

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$$\begin{cases} d\Delta u + f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

must be radially symmetric and $\partial u/\partial r < 0$ for $0 < r = |x| < 1$ where f is a locally Lipschitz continuous function and B is the unit ball centered at the origin O.

In [3–5], Berestycki and Nirenberg introduce the sliding method to prove the monotonicity in the whole of the domain. After that, the symmetry and monotonicity of solutions have attracted wide attention in the academic community, see [2,7–9,13,18]. Some authors have study the symmetry of the least-energy solution or solution with lower Morse index, their methods are heavily depending on the sign of the first Dirichlet eigenvalue of the corresponding linearized operator, see [10,15,17,22,23]. The main tool in the method of moving plane and sliding method is various kinds of maximum principle (see [5,24]), which allows us to compare different functions satisfying differential inequalities.

For problem (1.1) with boundary condition of Dirichlet or Neumann type, many researchers are focused on the least energy solution, a single-peak, multipeak spike-layer solution, see [11,16,19–21,26–28]. When the domain is a spherical sector, Berestycki and Pacella [7] prove the radial symmetry properties of positive solutions of (1.1) with mixed boundary conditions provided the amplitude of spherical sector is less or equal to π, and Zhu [29] proves the same result for singular solution when the amplitude is greater than π and under some supercritical growth conditions for smooth function f.

In this paper we will use the method of moving (rotating) plane to prove some symmetry and monotonicity results for positive solutions of semilinear elliptic equation under different mixed boundary conditions in a spherical cone.

We define the open cone $\mathcal{C} = \mathcal{C}_\alpha$ by $\mathcal{C}_\alpha = \{x \in \mathbb{R}^n : x_1 > |x|\cos(\alpha/2)\}$ and the spherical cone $\Sigma = \Sigma_\alpha$ with radius 1 and aperture α by

$$\Sigma_\alpha = B_1 \cap \mathcal{C}_\alpha = \{x \in \mathbb{R}^n : 0 < |x| = r < 1, x_1 > |x|\cos\frac{\alpha}{2}\}$$

and the boundary $\partial\Sigma$ of Σ consists of two parts Γ_D and Γ_N, where $\Gamma_D = \partial\Sigma \cap \partial\mathcal{C}$ and $\Gamma_N = \partial\Sigma\backslash\Gamma_D$. Here B_1 is the unit ball centered at the origin O.

Consider the following equation with mixed boundary condition

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Sigma, \\ u > 0 & \text{in } \Sigma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D \end{cases} \tag{1.2}$$

where ν is the unit outer normal to Σ.

The main results in this paper are as follows:

Theorem 1.1. *Let f be a locally Lipschitz continuous function and let $\alpha \in (0, 2\pi)$ if $n = 2$ and $\alpha \in (0, \pi]$ if $n > 2$. If $u \in C^2(\overline{\Sigma}\backslash\{O\}) \cap C(\overline{\Sigma})$ is a solution of (1.2), then we have that*

(1) u is axially symmetric with respect to x_1-axis and on each sphere $S_r := \{x : |x| = r\}$ with $0 < r \le 1$, u is strictly decreasing as the angle of $\overrightarrow{Ox}$ and positive x_1-axis. Moreover,

$$x_j \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_j} > 0 \text{ for } x_j > 0, j = 2, 3, \dots, n. \tag{1.3}$$