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Isometric weighted composition operators on weighted Bergman spaces ☆

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A B S T R A C T

We characterize the isometric weighted composition operators on weighted Bergman spaces over the unit disk. We also determine the Wold decomposition of isometric weighted composition operators acting on a class of general reproducing kernel Hilbert spaces in the case when the composition symbol has an interior fixed point, and characterize the numerical range of isometric weighted composition operators on weighted Bergman spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $\mathcal{H}(\mathbb{D})$ be the space of holomorphic, complex valued, functions on $\mathbb{D}$. For $u, \phi \in \mathcal{H}(\mathbb{D})$, with a nonconstant $\phi : \mathbb{D} \to \mathbb{D}$, the weighted composition operator (WCO) $W_{u,\phi}$ on $\mathcal{H}(\mathbb{D})$ is defined by

$$W_{u,\phi} f = u(f \circ \phi).$$

Choosing $\phi(z) = z$, the weighted composition operator $W_{u,\phi}$ becomes the multiplication operator M_u. In the case when $u \equiv 1$ on $\mathbb{D}$, $W_{u,\phi}$ is the composition operator C_ϕ on $\mathcal{H}(\mathbb{D})$.

In this paper we investigate the isometric weighted composition operators on spaces of holomorphic functions, and in particular on the weighted Bergman spaces over the unit disk.

For $\alpha > -1$, the weighted Bergman space $L^2_a(dm_\alpha)$ on the unit disk is defined as

$$L^2_a(dm_\alpha) = \{f \in \mathcal{H}(\mathbb{D}); \|f\|^2_\alpha = \int_{\mathbb{D}} |f(z)|^2 dm_\alpha(z) < \infty\},$$

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where m denotes the normalized Lebesgue area measure on $\mathbb{D}$ and

$$dm_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dm(z).$$

When $\alpha = 0$, we get the classical Bergman space $L_a^2(dm)$.

Weighted Bergman spaces are reproducing kernel Hilbert spaces with a positive definite kernel $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ given by

$$K^\alpha(w,z) = \frac{1}{(1-\overline{z}w)^{\alpha+2}}.$$

We will denote the corresponding point evaluation functions at z by K_z^α, and the normalized point evaluation functions by k_z^α. Since $\|K_z^\alpha\|^2 = K^\alpha(z,z) = \frac{1}{(1-|z|^2)^{\alpha+2}}$,

$$k_z^\alpha(w) = \frac{(1-|z|^2)^{\frac{\alpha}{2}+1}}{(1-\overline{z}w)^{\alpha+2}}.$$

The class of weighted composition operators is defined by using the natural operations that one can perform on spaces of functions. This class plays a particularly important role when determining the isometric operators on some Banach spaces of holomorphic functions. For example, Forelli determined in [7] that the isometries of the Hardy spaces H^p, $p \neq 2$, over the unit disk $\mathbb{D}$, are weighted composition operators. Similarly, Kolaski's results from [12] characterize the isometric operators on the Bergman spaces L_a^p, $p \neq 2$, over general Runge domains as weighted composition operators.

The cases when $p = 2$ represent the Hilbert space case, and so there are many other isometries acting on these spaces. Such are, for example, all of the unitaries that are defined by a simple change of basis. Still, the isometries (unitary, or not) that are also weighted composition operators are of particular interest also in the Hilbert space case, and are sometimes referred to as canonical isometries.

There is a vast literature dealing with the properties of weighted composition operators acting on a variety of spaces. Their boundedness and compactness on the Hardy spaces was determined in [3], and on the Bergman spaces in [5], [6] and [13]. The isometric weighted composition operators on the Hardy space H^2 were explored in [13] and [16]. The unitary weighted composition operators are of a particular interest, and they have been characterized even for some more general classes of reproducing kernel Hilbert spaces, including the Hardy and the Bergman spaces. See, for example, [14], [15], [18], and the references therein.

As it was shown in [14], a weighted composition operator on the weighted Bergman spaces is a co-isometry if and only if it is unitary. On the other hand, since for a unitary weighted composition operator on the Bergman spaces the composition symbol has to be a disk automorphism, the following simple example shows the existence of non-unitary isometric weighted composition operators.

Example. Let ϕ be a finite Blaschke product of degree n, and let $u(z) = \frac{1}{\sqrt{n}}\phi'(z)$. Then the weighted composition operator $W_{u,\phi}$ is isometric on the Bergman space $L_a^2(dm)$. This is easy to see since, using the change of variable formula and the fact that ϕ is of constant multiplicity n, we have that for any $f \in L_a^2(dm)$

$$\|W_{u,\phi}f\|^2 = \int_{\mathbb{D}} |f(\phi(z))|^2 \frac{1}{n}|\phi'(z)|^2 dm(z) = \int_{\mathbb{D}} |f(w)|^2 dm(w) = \|f\|^2.$$

Including the introduction, the paper contains three sections. Section 2 deals with the conditions that determine the isometric weighted composition operators acting on the weighted Bergman spaces over the unit disk.

Section 3 explores the Wold decomposition and some related properties of isometric weighted composition operators on more general reproducing kernel Hilbert spaces, and the characterization of the Wold