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Perturbations of superstable linear hyperbolic systems

Irina Kmit [a,b,*], Natalya Lyul'ko [c]

[a] Institute of Mathematics, Humboldt University of Berlin, Rudower Chaussee 25, D-12489 Berlin, Germany
[b] Institute for Applied Problems of Mechanics and Mathematics, Ukrainian National Academy of Sciences, Ukraine
[c] Sobolev Institute of Mathematics, Russian Academy of Sciences and Novosibirsk State University, Russia

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ABSTRACT

The paper deals with initial-boundary value problems for linear non-autonomous first order hyperbolic systems whose solutions stabilize to zero in a finite time. We prove that problems in this class remain exponentially stable in L^2 as well as in C^1 under small bounded perturbations. To show this for C^1, we prove a general smoothing result implying that the solutions to the perturbed problems become eventually C^1-smooth for any L^2-initial data.

1. Introduction

A linear system

$$\frac{d}{dt}x(t) = A(t)x(t), \quad x(t) \in X \quad (0 \leq t \leq \infty), \tag{1.1}$$

on a Banach space X is called *exponentially stable* if there exist positive reals γ and $M = M(\gamma)$ such that every solution $x(t)$ satisfies the estimate

$$\|x(t)\| \leq Me^{-\gamma t}\|x(0)\|, \quad t \geq 0, \tag{1.2}$$

where $\|\cdot\|$ denotes the norm in X.

The papers [2,3,8] address a stronger property of exponentially stable systems, known as superstability. They consider the Cauchy problem for the *autonomous* version of (1.1), where $A(t) = A$ does not depend

* Corresponding author.
E-mail addresses: kmit@mathematik.hu-berlin.de (I. Kmit), natlyl@mail.ru (N. Lyul'ko).

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on t. Moreover, $A : X \to X$ is supposed to be the infinitesimal generator of a strongly continuous semigroup $T(t)$; see [14,27]. A semigroup $T(t)$ is called *superstable* [2,3,22,29] if its *stability index* is $-\infty$, that is

$$\lim_{t\to\infty} \frac{\log \|T(t)\|}{t} = -\infty.$$

In this case the system (1.1) is called superstable also. The superstability property implies that the system is exponentially stable and, moreover, the estimate (1.2) holds for every $\gamma > 0$. For a superstable system the resolvent $R(\lambda; A)$ of the operator A, which is defined by the formula

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt, \quad x \in X,$$

is an entire function of the complex parameter λ, and the spectrum of the operator A, which we denote by $\sigma(A)$, is empty. The superstability property makes sense only for systems in infinite-dimensional Banach spaces, since any linear operator $A : X \to X$ in a finite-dimensional space X has a non-empty point spectrum, and the stability index is equal to the maximum of the real parts of the eigenvalues of A.

An important subclass of superstable systems, that will be studied in the present paper, consists of the systems whose solutions stabilize to zero after some time. The time of the stabilization is called a *finite time extinction*. The simplest example is given by the initial-boundary value problem [14]

$$\begin{aligned}
&u_t + u_x = 0, && (x,t) \in (0,1) \times (0,\infty),\\
&u(x,0) = u_0(x), && x \in [0,1],\\
&u(0,t) = 0, && t \in (0,\infty).
\end{aligned}$$

It is easy to check that all solutions to this problem stabilize to zero for $t > 1$. Similar examples for the wave equation are given in [7,19,24].

Here we address superstable initial-boundary value problems with finite time extinction for linear *non-autonomous* hyperbolic systems. We consider bounded perturbations of such problems and investigate the asymptotic behavior of their solutions. In contrast to the autonomous case, which is well-studied, the non-autonomous case has been considered in the literature only episodically.

The recent papers [25,26,28,31–33] are devoted to superstable hyperbolic models intensively used in the control theory. By introducing control parameters in the boundary conditions and/or in the coefficients of the differential equations, such systems can be stabilized to a desired state in a finite time, which, from the physical point of view, is even more preferable than the infinite time stabilization. Superstable hyperbolic systems are usually supplemented with the so-called quiet boundaries [32], where the influence of the reflected waves is minimized or even neglected. Mathematically, the quiet boundaries are described by means of the so-called *smoothing boundary conditions*, that also will be considered in the present paper.

The paper is organized as follows. In Section 2 we state the problem and formulate our results about the existence of evolution families, smoothing properties of solutions, and exponential stability of solutions. Some comments and examples related to applications are given in Section 3. In particular, Examples 3.4 and 3.5 show how our results obtained for linear systems can be applied to show the exponential stability of solutions to nonlinear problems. Using a priori estimates, in Section 4 we prove that the problem under consideration generates an evolution family on $L^2(0,1)^n$. In Section 4.3 we extend the results of [10,16,17,21] by showing that boundary operators of reflection type cause the smoothing effect in the sense that the solutions reach the C^1-regularity for any L^2-initial data after some time. Furthermore, we provide general conditions for that. We also discuss the relationship between the smoothing property and the stabilization to zero. Finally, in Section 5.2, using the variation of constants formula, we prove that superstable hyperbolic operators remain exponentially stable under small bounded perturbations.

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