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Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution ☆

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ABSTRACT

The article introduces a new general solution to a family of loaded ordinary differential equations and discusses its properties. It provides necessary and sufficient conditions for the well-posedness of a linear nonlocal boundary value problem for a system of loaded hyperbolic equations with mixed derivatives. Algorithms for solving the boundary value problems for loaded differential equations are proposed.

1. Introduction

Loaded differential equations, also known as differential boundary equations, play an important role in modeling various processes of natural sciences. In [16,17], they describe problems of long-term forecasting and control of groundwater level in soil moisture. Various problems for loaded differential equations and methods for solving these problems are studied in [1–3,15–17].

In this paper, we consider the following nonlocal boundary value problem for the system of loaded hyperbolic equations with mixed derivatives in the domain $\bar{\Omega} = [0, \omega] \times [0, T]$:

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u}{\partial x} + B(x,t)\frac{\partial u}{\partial t} + C(x,t)u + \sum_{j=1}^{m+1}\Big[K_{2j}(x,t)\frac{\partial u(x,\theta_{j-1})}{\partial x} +$$

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$$+K_{1j}(x,t)\frac{\partial u(x,t)}{\partial t}\Big|_{t=\theta_{j-1}}+K_{0j}(x,t)u(x,\theta_{j-1})\Big]+f(x,t), \qquad (x,t)\in\bar{\Omega}, \quad u\in R^n, \tag{1.1}$$

$\theta_0=0<\theta_1<\theta_2<\ldots<\theta_{m-1}<\theta_m=T$,

$$P_2(x)\frac{\partial u(x,0)}{\partial x}+P_1(x)\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}+P_0(x)u(x,0)+S_2(x)\frac{\partial u(x,T)}{\partial x}+$$

$$+S_1(x)\frac{\partial u(x,t)}{\partial t}\Big|_{t=T}+S_0(x)u(x,T)=\varphi(x), \qquad x\in[0,\omega], \tag{1.2}$$

$$u(0,t)=\psi(t), \qquad t\in[0,T]. \tag{1.3}$$

Here the $(n\times n)$ matrices $A(x,t)$, $B(x,t)$, $C(x,t)$, $K_{ij}(x,t)$, $i=0,1,2$, $j=\overline{1,m+1}$, and the n vector $f(x,t)$ are continuous on $\bar{\Omega}$, the $(n\times n)$ matrices $P_i(x)$, $S_i(x)$, $i=0,1,2$, and the n vector $\varphi(x)$ are continuous on $[0,\omega]$, and the n vector $\psi(t)$ is continuously differentiable on $[0,T]$. The norms of n vector $u=(u_i)$ and $(n\times n)$ matrix $A=(a_{ik})$, $i,k=\overline{1,n}$, are defined as $||u||=\max\limits_{i=\overline{1,n}}|u_i|$ and $||A||=\max\limits_{i=\overline{1,n}}\sum\limits_{k=1}^{n}|a_{ik}|$.

Loaded differential equations can also be obtained by replacing the integral terms of integro-differential equations by approximate formulas.

There are other sources (see [4–8,10,12–14,11] and references cited therein) more focused on nonlocal boundary value problems for systems of hyperbolic equations (without loaded terms).

We introduce the following spaces:

$C(\bar{\Omega},R^n)$ is the space of continuous functions $u:\bar{\Omega}\to R^n$ with the norm $||u||_0=\max\limits_{(x,t)\in\bar{\Omega}}||u(x,t)||$,

$C([0,\omega],R^n)$ is the space of continuous functions $\varphi:[0,\omega]\to R^n$ with the norm $||\varphi||_0=\max\limits_{x\in[0,\omega]}||\varphi(x)||$,

$C^1([0,T],R^n)$ is the space of continuously differentiable functions $\psi:[0,T]\to R^n$ with the norm $\|\psi\|_1=\max(\max\limits_{t\in[0,T]}\|\psi(t)\|,\max\limits_{t\in[0,T]}\|\dot{\psi}(t)\|)$.

For a function $u(x,t)\in C(\overline{\Omega},R^n)$, we set $\|u(x,\cdot)\|_0=\max\limits_{t\in[0,T]}\|u(x,t)\|$.

A function $u(x,t)\in C(\bar{\Omega},R^n)$ that has the partial derivatives $\frac{\partial u(x,t)}{\partial x}\in C(\bar{\Omega},R^n)$, $\frac{\partial u(x,t)}{\partial t}\in C(\bar{\Omega},R^n)$, and $\frac{\partial^2 u(x,t)}{\partial x\partial t}\in C(\bar{\Omega},R^n)$ is called a classical solution to problem (1.1)–(1.3) if for all $(x,t)\in\bar{\Omega}$ it satisfies the loaded differential equation (1.1) and boundary conditions (1.2), (1.3). Here and below in the article, we assume that the observed functions at the boundary of the domain and at the end-points of the interval have one-sided derivatives.

The loaded terms of equation (1.1), terms with coefficients $K_{ij}(x,t)$, $i=0,1,2$, $j=\overline{1,m+1}$, essentially affect qualitative properties of the equation and the problems for it.

Consider the Goursat problem for the loaded hyperbolic equation

$$\frac{\partial^2 u}{\partial x\partial t}=2\frac{\partial u(x,0.5)}{\partial x}+1, \qquad (x,t)\in[0,1]\times[0,1], \tag{1.4}$$

$$u(0,t)=0, \qquad t\in[0,1], \qquad u(x,0)=x, \qquad x\in[0,1]. \tag{1.5}$$

It is well-known that the Goursat problem for hyperbolic equations with mixed derivatives has a unique solution. But problem (1.4), (1.5) has no solutions. Indeed, if a function $u^*(x,t)$ is a solution to problem (1.4), (1.5), then the function $v^*(x,t)=\frac{\partial u^*(x,t)}{\partial x}$ is a solution to the Cauchy problem for the family of loaded ordinary differential equations

$$\frac{\partial v}{\partial t}=2v(x,0.5)+1, \qquad t\in[0,T], \quad x\in[0,1], \tag{1.6}$$

$$v(x,0)=1, \qquad x\in[0,1]. \tag{1.7}$$