J. Math. Anal. Appl. 461 (2018) 837–867

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Nonlocal operators of order near zero

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A R T I C L E I N F O

Article history:
Received 26 July 2017
Available online 8 December 2017
Submitted by V. Radulescu

Keywords:
Dirichlet forms
Integral operators
Nonlocal equations
Compact embeddings

A B S T R A C T

We study Dirichlet forms defined by nonintegrable Lévy kernels whose singularity at the origin can be weaker than that of any fractional Laplacian. We show some properties of the associated Sobolev type spaces in a bounded domain, such as symmetrization estimates, Hardy inequalities, compact inclusion in L^2 or the inclusion in some Lorentz space. We then apply those properties to study the associated nonlocal operator $\mathfrak{L}$ and the Dirichlet and Neumann problems related to the equations $\mathfrak{L}u = f(x)$ and $\mathfrak{L}u = f(u)$ in Ω.

1. Introduction

The aim of this paper is to study the properties of the bilinear Dirichlet form associated to a kernel $J : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ and a given bounded set $\Omega \subset \mathbb{R}^N$, defined by

$$\mathcal{E}(u, v) = \frac{1}{2} \iint_{Q_\Omega} (u(x) - u(y))(v(x) - v(y))J(x, y)\, dxdy, \tag{1.1}$$

where

$$Q_\Omega = (\Omega^c \times \Omega^c)^c,$$

and J is a measurable function satisfying

$$\begin{cases} J(x,y) \geq 0, \quad J(x,y) = J(y,x), \\ \sup\limits_{x\in\mathbb{R}^N} \displaystyle\int_{\mathbb{R}^N} \min(1, |x-y|^2)J(x,y)\, dy < \infty. \end{cases} \tag{H$_0$}$$

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https://doi.org/10.1016/j.jmaa.2017.12.011
0022-247X/

Condition (H_0) means that J is a Lévy type kernel. We also assume that the kernel lies in the so-called nonintegrable side, that is,

$$J(x,y) \geq \mathcal{K}(x-y) \geq 0, \qquad \mathcal{K} \notin L^1(B_\varepsilon) \quad \forall\, \varepsilon > 0, \tag{H_0'}$$

where $B_\varepsilon = \{z \in \mathbb{R}^N, |z| < \varepsilon\}$. Both these hypotheses (H_0)–(H_0') are assumed throughout the paper without further mention.

The power case $\mathcal{K}(z) = |z|^{-N-\alpha}$ for some $0 < \alpha < 2$ is well known and is related to stable processes, see also the associated operator, the fractional Laplacian, below. We are mostly interested in the weakly singular case which separates the fractional Laplacian side to the integrable side, i.e., $\lim_{z\to 0} |z|^{N+\alpha}\mathcal{K}(z) = 0$ for every $\alpha > 0$. One of the most important examples of such kind of operators is the one corresponding to the so-called geometric stable process, with kernel $K(z) \sim |z|^{-N}$ for $|z| \sim 0$, see [4,30], but we also include possible logarithmic perturbations of those kernels.

We consider the spaces

$$\mathcal{H}_J(\Omega) = \left\{u : \mathbb{R}^N \to \mathbb{R},\ u|_\Omega \in L^2(\Omega),\, \mathcal{E}(u,u) < \infty\right\} \tag{1.2}$$

and

$$\mathcal{H}_{J,0}(\Omega) = \{u \in \mathcal{H}_J(\Omega),\, u \equiv 0 \text{ in } \Omega^c\}\,, \tag{1.3}$$

with norm

$$\|u\|_{\mathcal{H}_J} = \left(\int\limits_\Omega u^2 + \mathcal{E}(u,u)\right)^{1/2}.$$

Hypothesis (H_0) implies

$$H_0^1(\Omega) \subset \mathcal{H}_{J,0}(\Omega) \subset \mathcal{H}_J(\Omega) \subset L^2(\Omega),$$

if we consider the functions in $H_0^1(\Omega)$ extended by zero outside Ω. In the fractional Laplacian case $\mathcal{K}(z) = |z|^{-N-\alpha}$ for some $0 < \alpha < 2$ (and $N > \alpha$), it holds

$$\mathcal{H}_J(\Omega) \subset H^{\alpha/2}(\Omega) \subset L^{\frac{2N}{N-\alpha}}(\Omega),$$

thanks to the Hardy–Sobolev inequality, where $H^{\alpha/2}(\Omega)$ is the usual fractional Sobolev space of functions in $L^2(\Omega)$ such that $\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\, dxdy < \infty$. As a byproduct we have $\mathcal{H}_J(\Omega) \hookrightarrow L^2(\Omega)$ compactly.

On the very other hand, in the case of integrable kernels, $\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} J(x,y)\, dy = B < \infty$ (thus not satisfying (H_0')), we have $\mathcal{E}(u,u) \leq B\|u\|_2^2$ and therefore $\mathcal{H}_{J,0}(\Omega) \equiv L^2(\Omega)$.

A main objective of this paper is to establish the exact place where $\mathcal{H}_{J,0}(\Omega)$ and $\mathcal{H}_J(\Omega)$ lie in relation to $L^2(\Omega)$. We prove that if $\mathcal{K}$ does not oscillate too much at the origin then the inclusion $\mathcal{H}_{J,0}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. If in addition $\lim_{z\to 0} |z|^N\mathcal{K}(z) = \infty$, then also $\mathcal{H}_J(\Omega) \hookrightarrow L^2(\Omega)$ is compact. See Theorems 2.1 and 2.2.

The compactness of the inclusion $\mathcal{H}_{J,0}(\Omega) \hookrightarrow L^2(\Omega)$ can be explained by the sharper inclusion into some Lorentz space

$$\mathcal{H}_{J,0}(\Omega) \subset \mathcal{L}_{\mathcal{A},2}(\Omega) \subsetneq L^2(\Omega),$$