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The 2D regularized incompressible Boussinesq equations with general critical dissipations

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ABSTRACT

Considering the 2D regularized Boussinesq equations with fractional dissipations $(\Lambda^\alpha u, \Lambda^\beta \theta)$ and convection terms $(\Lambda^{-\gamma} u \cdot \nabla u, \Lambda^{-\gamma} u \cdot \nabla\theta)$, where $\Lambda = \sqrt{-\Delta}$ and $\gamma \geq 0$, we prove the global existence and uniqueness of the solution in two critical cases. The first case has fractional dissipations $(\Lambda^\alpha u, \Lambda^\beta \theta)$, where $\alpha + \beta = 1 - \gamma$, $\beta > 0$, and the second one has particular dissipation $(\Lambda^{1-\gamma} u, 0)$. In particular, for the case $\gamma = 0$, we give some decay estimates for (θ, u) and the uniform estimate for G independent of time, where $G = \partial_1 u_2 - \partial_2 u_1 - \partial_1 \Lambda^{-\alpha}\theta$.

1. Introduction

The standard 2D Boussinesq equations with dissipations are

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p + \theta e_2, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\ \nabla \cdot u = 0, \\ (u, \theta) \mid_{t=0} = (u_0, \theta_0), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$, p and θ represent velocity, pressure and density respectively, e_2 is the unit vector in the vertical direction, and ν and κ are nonnegative dissipation parameters. When $\nu, \kappa > 0$, the global regularity result can be easily obtained as for the 2D incompressible Navier–Stokes equations. But the global well-posedness theorem is open when $\nu = \kappa = 0$, expect in the case $\theta = 0$. The standard Boussinesq equations, which model geophysical flows such as atmospheric fronts and oceanic circulation, play an important role in Rayleigh–Bénard convection. Mathematically, the 2D Boussinesq equations, which serve as a lower-dimensional model of the 3D hydrodynamics equations, retain the vortex stretching mechanism as in the case of the 3D incompressible Navier–Stokes equations and the 3D incompressible Euler equations. The in-

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viscid Boussinesq equations can be identified with the 3D incompressible Euler equations for axisymmetric flows. See [20,21,24] for more details.

Recently, mathematicians began to study the 2D Boussinesq equations with fractional dissipations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ (u, \theta) \mid_{t=0} = (u_0, \theta_0), \end{cases} \tag{1.2}$$

where $(\alpha, \beta) \in [0, 2]^2$ and u, p, θ, ν and κ are defined similarly to the case of (1.1). For convenience, we adopt the convention that ν can be understood as a continuous function of α, satisfying $\nu(\alpha) > 0$ when $\alpha \in (0, 2]$ and $\nu(0) = 0$. Similarly, κ can be understood as a continuous function of β, satisfying $\kappa(\beta) > 0$ when $\beta \in (0, 2]$ and $\beta(0) = 0$. That means there is no diffusion term $\nu\Lambda^\alpha u$ ($\kappa\Lambda^\beta\theta$) when $\alpha = 0$ ($\beta = 0$). For convenience, we use constants (ν, κ) instead of $(\nu(\alpha), \kappa(\beta))$ when α and β are positive. According to the introduction in [22], $\alpha + \beta = 1$ is referred to as the critical case, $\alpha + \beta > 1$ is referred to as the subcritical case and $\alpha + \beta < 1$ is referred to as the supercritical case. We list some recent results next.

1.1. *Subcritical cases*

(1) Particular cases. Chae [4] and Hou and Li [16] proved that there exists a unique global solution for the system (1.2) with $(\alpha, \beta) = (2, 0)$ or $(\alpha, \beta) = (0, 2)$.

(2) General cases ($\nu, \kappa > 0$). Miao and Xue [22] obtained similar results for the system (1.2) when α, β satisfy

$$\frac{6-\sqrt{6}}{4} < \alpha < 1,\ 1-\alpha < \beta < \min\{\frac{7+2\sqrt{6}}{5}\alpha - 2, \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha}, 2-2\alpha\}.$$

Constantin and Vicol [8] obtained the global well-posedness theorem when α, β satisfy

$$0 < \alpha < 2,\ \frac{2}{2+\alpha} < \beta < 2.$$

Recently, Ye and Xu [30] proved a similar result with a new range of fractional power

$$0.7351 \approx \frac{10-2\sqrt{10}}{5} < \alpha < 1,\ 1-\alpha < \beta < \min\{3-3\alpha, \frac{\alpha}{2}, \frac{3\alpha^2+4\alpha-4}{8(1-\alpha)}\}.$$

See [30] for more results concerning subcritical cases.

1.2. *Critical cases*

(1) Particular cases. Hmidi et al. [14,15], who introduced a combined quantity of the vorticity and the Riesz transform of the density, obtained the global well-posedness theorem when $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$.

(2) General cases. Jiu et al. [17] first obtained the global well-posedness theorem for the system (1.2) with the initial data $(u_0, \theta_0) \in B^\sigma_{2,1}(\mathbb{R}^2) \times B^2_{2,1}(\mathbb{R}^2)$ $(\sigma \geq \frac{5}{2})$ when α, β satisfy

$$\alpha + \beta = 1,\ \alpha_0 < \alpha < 1,\ \alpha_0 = \frac{23-\sqrt{145}}{12} \approx 0.9132.$$

They gave the detailed proof of the result with the initial data $(u_0, \theta_0) \in (B^{2+\beta}_{2,1}(\mathbb{R}^2), B^2_{2,1}(\mathbb{R}^2))$.

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