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The boundary control method and de Branges spaces. Schrödinger equation, Dirac system and discrete Schrödinger operator

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ABSTRACT

We establish the relationship between the Boundary Control method for dynamic inverse problems and the method of de Branges on the examples of dynamical systems for Schrödinger and Dirac operators on a half-line and semi-infinite discrete Schrödinger operator. For each of the system we construct the de Branges space and describe in natural dynamic terms its attributes: the set of function the space consists of, the scalar product, the reproducing kernel.

1. Introduction

In [3,14] the authors attempted to look at different approaches to inverse problems for a Schrödinger operator on a half-line from one (dynamic) point of view. They have shown that Gelfand–Levitan [21], Krein [26], Simon [36] and Remling [33] equations can be derived within the framework of the Boundary Control (BC) method.

In [33,32] the author answering questions about A-function posed by Simon in [36,22], used the de Branges method and de Branges spaces. In [3] it is shown that the equations derived by Remling [32] are in fact Krein-type equations and they have clear dynamic interpretation in terms of the BC method. Relationships of the BC method and the de Branges method with functional models of symmetric operator are considered in [10,12,15] and [35]. In the present paper we will be dealing with the inverse problems theory: we would like to elaborate and clarify the correspondence between the BC method and de Branges method on a

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basis of three dynamical systems, our aim will be to show that all ingredients of a de Branges space: set of functions the space consist of, scalar product, reproducing kernel, have their natural dynamic counterparts which are central objects in the BC method. We note that the inverse dynamic problem for general type canonical system have not been considered in literature by the present moment. Our future goal will be to work out a version of the BC method for the corresponding inverse problem. This paper supposed to be the first one in the series. Another possible application of methods discussed here is an inverse problem for a Krein string. In [25,26] certain procedure of recovering of a string was formulated, but no theorems were proved, and no correspondence between classes of inverse data and strings (densities) was established. We hope that our approach that clarify a relationship between dynamic (BC) and spectral (de Branges) methods, will stimulate a progress in the Krein method, which is "in between" spectral and dynamic.

The BC method was originally proposed for solving the boundary inverse problem for the multidimensional wave equation [6,5,7], but since then it has been applied to all main types of linear equations of mathematical physics, inverse problems on graphs, spectral estimation and identification problems, see [7–9,11,23,24] and references therein. This method uses the deep relationships between inverse problems of mathematical physics, functional analysis, geometric optic and control theory for partial differential equations. This approach has several advantages: it is linear, it is applicable to a wide range of one-dimensional, multi-dimensional and discrete inverse problems; it can identify coefficients occurring in highest order terms; it lends itself to straightforward *linear* algorithmic implementations; it is local: on a larger interval we know the inverse data, on a larger interval we can recover the coefficients. The development of the BC method showed that a number of facts in the spectral theory were given a transparent dynamical interpretation [4,29,31], and the connection of the BC method with the de Branges method is yet another example.

In our approach we deal with dynamical systems with boundary control. Fixing time $T > 0$ and varying the controls, we take the set of states of the system at this time (the reachable set), taking the Fourier image of this set yields the new space. We equip this space with the norm generated by so-called *connecting operator* to get a Hilbert space of analytic functions. Then we construct the reproducing kernel in this space by solving the Krein-type equations and use the theorem of de Branges [17,20] to show that this space is in fact a de Branges space. We develop this approach on the basis of initial boundary value problem for three systems: wave equation with a potential, Dirac system on a half-line and dynamical system with discrete time for semi-infinite discrete Schrödinger operator.

In the second section we provide all necessary information on de Branges spaces following [32] and [34]. In the third section we deal with a Schrödinger operator on a half-line, the fourth and fifth sections are devoted to a Dirac operator on a half-line and a semi-infinite discrete Schrödinger operator. For each operator we set up the forward and the inverse dynamic problems, introduce the dynamic inverse data and operators of the Boundary Control method. Then for each system we introduce the special space of functions which (as we prove) will be a de Branges space.

2. de Branges spaces

Here we provide the information on de Branges spaces in accordance with [32,34]. The entire function $E : \mathbb{C} \mapsto \mathbb{C}$ is called a *Hermite–Biehler function* if $|E(z)| > |E(\overline{z})|$ for $z \in \mathbb{C}_+$. We use the notation $F^{\#}(z) = \overline{F(\overline{z})}$. The *Hardy space* H_2 is defined by: $f \in H_2$ if f is holomorphic in $\mathbb{C}^+$ and $\sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)|^2\, dx < \infty$. Then the *de Branges space* $B(E)$ consists of entire functions such that:

$$B(E) := \left\{ F : \mathbb{C} \mapsto \mathbb{C},\ F \text{ entire}, \int\limits_{\mathbb{R}} \left| \frac{F(\lambda)}{E(\lambda)} \right|^2 d\lambda < \infty,\ \frac{F}{E}, \frac{F^{\#}}{E} \in H_2 \right\}.$$