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A Matsumoto–Yor characterization for Kummer and Wishart random matrices

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A R T I C L E I N F O

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In the paper we resolve positively the conjecture on a characterization of matrix Kummer and Wishart laws through independence property, which was posed in Koudou (2012) [12]. Apart from the probabilistic result, we determine the general solution of the functional equation associated to the characterization problem under weak assumptions.

1. Introduction

Bernstein [2] proved, under some technical assumptions, that if random variables X and Y are independent, then random variables $U = X + Y$ and $V = X - Y$ are independent if and only if X and Y are Gaussian. Under additional assumptions that X and Y have densities, this result is equivalent to finding the general solution of the following functional equation

$$f_X(x)f_Y(y) = 2f_U(x+y)f_V(x-y), \qquad \text{a.e. } (x,y) \in \mathbb{R}^2,$$

where f_X, f_Y, f_U and f_V are unknown densities (thus measurable and a.e. non-negative) of respective random variables. Under mild regularity conditions, this functional equation has a solution

$$f_X(x) = \exp\{Ax^2 + B_1x + C_1\}, \qquad f_U(x) = \exp\{\tfrac{1}{2}Ax^2 + \tfrac{1}{2}(B_1+B_2)x + C_3\},$$
$$f_Y(x) = \exp\{Ax^2 + B_2x + C_2\}, \qquad f_V(x) = \exp\{\tfrac{1}{2}Ax^2 + \tfrac{1}{2}(B_1-B_2)x + C_4\},$$

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for a.e. $x \in \mathbb{R}$, for some $A, B_i, C_i \in \mathbb{R}$ with $C_1 + C_2 = C_3 + C_4 + \log(2)$. Since f_X, f_Y, f_U and f_V have to be integrable on $\mathbb{R}$, A has to be strictly negative so that X and Y are Gaussian with the same variance.

Many similar examples of the so-called independence characterizations have been identified throughout the years. They follow the following general scheme: determine the distributions of X and Y if they are independent and the components of $(U, V) = \psi(X, Y)$ are independent, where ψ is some given function defined on the support of (X, Y). If ψ is a diffeomorphism, under the assumption of existence of densities of X and Y, such problem is equivalent to solving the following associated functional equation

$$f_X(x)f_Y(y) = J(x, y)f_U(\psi_1(x, y))f_V(\psi_2(x, y)), \qquad \text{a.e. } (x, y) \in \mathbb{R}^2, \tag{1}$$

where $J = \left|\frac{\mathrm{d}\psi_1}{\mathrm{d}x}\frac{\mathrm{d}\psi_2}{\mathrm{d}y} - \frac{\mathrm{d}\psi_1}{\mathrm{d}y}\frac{\mathrm{d}\psi_2}{\mathrm{d}x}\right|$ is the Jacobian of ψ. Obviously, not every diffeomorphism yields a probabilistic solution (in which we are here particularly interested) and it is still not clear how to choose appropriate ψ.

There is a very interesting family of ψ's which was introduced in [14]. From the probabilistic point of view, this family is somehow related to the so-called Matsumoto–Yor property. Koudou and Vallois in [14] (see also [13]) considered $\psi^{(f)}(x, y) = (f(x + y), f(x) - f(x + y))$ where f is some regular function and asked the following question: for which f there exist independent X and Y such that U and V are independent. The classical Matsumoto–Yor property (see [24,25]) is obtained for $f^{(1)}(x) = x^{-1}$. If $f^{(2)}(x) = \log(x)$ we obtain the so-called Lukacs property (see [17,1,15,18,4]). Another important case identified in [14] was $f^{(3)}(x) = \log(1 + x^{-1})$, which is the subject of present study.

All above-mentioned cases have their matrix-variate analogues, where X and Y are considered to be random matrices (or equivalently, X and Y to be random variables valued in the set of matrices). Such properties are usually much harder to prove due to additional noncommutativity of matrix multiplication, which is not witnessed if X and Y are 1-dimensional. A generalization of the classical Matsumoto–Yor property was considered in [25,16,11] along with a characterization of probability laws having this property. This characterization was proven via solving the associated functional equation

$$A(\mathbf{x}) + B(\mathbf{y}) = C(\mathbf{x} + \mathbf{y}) + D\left(\mathbf{x}^{-1} - (\mathbf{x} + \mathbf{y})^{-1}\right), \qquad (\mathbf{x}, \mathbf{y}) \in \mathcal{S}_+^2, \tag{2}$$

where $\mathcal{S}_+$ is the cone of positive definite real matrices of full rank and $A, B, C, D\colon \mathcal{S}_+ \to \mathbb{R}$ are unknown functions. This functional equation is a matrix-variate version of (1) after taking the logarithm of both sides, which is allowed if one assumes that f_X and f_Y are positive on $\mathcal{S}_+$. Eq. (2) was solved in [11] under the assumption that A and B are continuous, which is the best result available at the moment. Actually, (2) was considered there in a more general setting, that is, on symmetric cones of which $\mathcal{S}_+$ is the prime example.

A generalization of other properties identified in [14] to matrices is not automatic and this is due to the fact there is no natural notion of division of matrices. For example, Lukacs property on $\mathcal{S}_+$ is stated with $\psi^{(2)}(\mathbf{x}, \mathbf{y}) = \left(\mathbf{x} + \mathbf{y}, g(\mathbf{x} + \mathbf{y}) \cdot \mathbf{x} \cdot g(\mathbf{x} + \mathbf{y})^\top\right)$, where $g\colon \mathcal{S}_+ \mapsto M_r$ is such that $g(\mathbf{x}) \cdot \mathbf{x} \cdot g(\mathbf{x})^\top = \mathrm{I}$ for any $\mathbf{x} \in \mathcal{S}_+$. Here "$\cdot$" denotes the ordinary matrix multiplication, I is the identity matrix in $\mathcal{S}_+$ and $\mathbf{x}^\top$ denotes the transpose of $\mathbf{x}$. One can take for example $g(\mathbf{x}) = (\mathbf{x}^{1/2})^{-1}$, where $\mathbf{x}^{1/2}$ is the unique positive definite square root of $\mathbf{x} = \mathbf{x}^{1/2} \cdot \mathbf{x}^{1/2}$. A characterization of laws having Lukacs property was considered in [3,6,8], and in the latter paper the general solution of the following associated functional equation was found

$$A(\mathbf{x}) + B(\mathbf{y}) = C(\mathbf{x} + \mathbf{y}) + D\left(g(\mathbf{x} + \mathbf{y}) \cdot \mathbf{x} \cdot g(\mathbf{x} + \mathbf{y})^\top\right), \qquad (\mathbf{x}, \mathbf{y}) \in \mathcal{S}_+^2$$

under the assumption that A and B are continuous.

In the present paper we will consider a generalization of $f^{(3)}$ to $\mathcal{S}_+$, which was introduced in [12]:

$$\psi^{(3)}(\mathbf{x}, \mathbf{y}) = \left(\mathbf{x} + \mathbf{y}, (\mathrm{I} + (\mathbf{x} + \mathbf{y})^{-1})^{1/2} \cdot (\mathrm{I} + \mathbf{x}^{-1})^{-1} \cdot (\mathrm{I} + (\mathbf{x} + \mathbf{y})^{-1})^{1/2}\right).$$

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