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A note on the Ostrovsky equation in weighted Sobolev spaces

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A B S T R A C T

In this work we consider the initial value problem (IVP) associated to the Ostrovsky equations

$$\left.\begin{aligned} u_t + \partial_x^3 u \pm \partial_x^{-1} u + u\partial_x u = 0, \qquad & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \qquad & \end{aligned}\right\}$$

We study the well-posedness of the IVP in the weighted Sobolev spaces

$$\dot{Z}_{s,\frac{s}{2}} := \{f \in H^s(\mathbb{R}) : \partial_x^{-1} f \in L^2(\mathbb{R})\} \cap L^2(|x|^s dx),$$

with $\frac{3}{4} < s \leq 1$.

1. Introduction

In this note we consider the initial value problem (IVP) associated to the Ostrovsky equations,

$$\left.\begin{aligned} u_t + \partial_x^3 u \pm \partial_x^{-1} u + u\partial_x u = 0, \qquad & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \qquad & \end{aligned}\right\} \tag{1.1}$$

The operator ∂_x^{-1} in the equations denotes a certain antiderivative with respect to the spatial variable x defined through the Fourier transform by $(\partial_x^{-1} f)\widehat{\ } := \frac{\widehat{f}(\xi)}{i\xi}$.

These equations were deducted in [20] as a model for weakly nonlinear long waves, in a rotating frame of reference, to describe the propagation of surface waves in the ocean. The sign of the third term of the equation is related to the type of dispersion.

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Linares and Milanés [16] proved that the IVP (1.1) for both equations is locally well-posed (LWP) for initial data u_0 in Sobolev spaces $H^s(\mathbb{R})$, with $s > \frac{3}{4}$, and such that $\partial_x^{-1}u_0 \in L^2(\mathbb{R})$. This result was obtained by the use of certain regularizing effects of the linear part of the equation. In [8] and [9] Isaza and Mejía used Bourgain spaces and the technique of elementary calculus inequalities, introduced by Kenig, Ponce, and Vega in [14], to prove local well-posedness in Sobolev spaces $H^s(\mathbb{R})$, with $s > -\frac{3}{4}$, for both equations. Furthermore Isaza and Mejía, in [10], established that the IVP (1.1), for both equations, is not quantitatively well-posed in $H^s(\mathbb{R})$ with $s < -\frac{3}{4}$. Recently, Li et al. in [15] proved that the IVP (1.1) with the minus sign is LWP in $H^{-3/4}(\mathbb{R})$.

In [12], Kato studied the IVP for the generalized KdV equation in several spaces, besides the classical Sobolev spaces. Among them, Kato considered weighted Sobolev spaces.

In this work we will be concerned with the well-posedness of the IVP (1.1) in weighted Sobolev spaces. This type of spaces arises in a natural manner when we are interested in determining if the Schwartz space is preserved by the flow of the evolution equations in (1.1). These spaces also appear in the study of the persistence in time of the regularity of the Fourier transform of the solutions of the IVP (1.1).

Some relevant nonlinear evolution equations as the KdV equation, the non-linear Schrödinger equation, the Benjamin–Ono equation and the Zakharov–Kuznetsov equation have also been studied in the context of weighted Sobolev spaces (see [3], [1], [4], [6], [7], [11], [17], [18], [19] and [2] and references therein).

We will study real valued solutions of the IVP (1.1) in the weighted Sobolev spaces

$$\dot{Z}_{s,r} := \{f \in H^s(\mathbb{R}) : \partial_x^{-1}f \in L^2(\mathbb{R})\} \cap L^2(|x|^{2r}dx),$$

with $s, r \in \mathbb{R}$.

The Ostrovsky equations are perturbations of the KdV equation with the nonlocal term $\pm\partial_x^{-1}u$. It is interesting to know how this term behaves when we are working in the context of weighted Sobolev spaces.

The relation between the indices s and r for the solutions of the IVP (1.1) can be found, following the considerations, contained in the work of Kato (for more details see [2]): it turns out that the natural weighted Sobolev space to study the IVP (1.1) is $\dot{Z}_{s,s/2}$.

Our aim in this article is to prove that the IVP (1.1) is locally well posed (LWP) in $\dot{Z}_{s,s/2}$ for $\frac{3}{4} < s \leq 1$. Our method of proof is based on the contraction mapping principle and has two ingredients. First of all, we use the result of local well posedness, obtained by Linares and Milanés in $X_s := \{f \in H^s(\mathbb{R}) : \partial_x^{-1}f \in L^2(\mathbb{R})\}$, with $s > \frac{3}{4}$. The statement of this result is as follows.

Theorem A. *Let $u_0 \in X_s$, $s > \frac{3}{4}$. Then there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution u of the IVP (1.1) such that*

$$u \in C([0,T]; X_s)\,, \tag{1.2}$$

$$\|\partial_x u\|_{L^4_T L^\infty_x} < \infty\,, \tag{1.3}$$

$$\|D^s_x \partial_x u\|_{L^\infty_x L^2_T} < \infty\,, \quad \text{and} \tag{1.4}$$

$$\|u\|_{L^2_x L^\infty_T} < \infty\,. \tag{1.5}$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in X_s such that the map datum-solution $\tilde{u}_0 \mapsto \tilde{u}$ is Lipschitz from V into the class defined by (1.2)–(1.5) with T' instead of T.

On the other hand, we need a tool to treat fractional powers of $|x|$. A key idea in this direction is to use a characterization of the generalized Sobolev space

$$L^p_b(\mathbb{R}^n) := (1-\Delta)^{-b/2}L^p(\mathbb{R}^n), \tag{1.6}$$

due to Stein (see [21] and [22]) (when $p = 2$, $L^2_b(\mathbb{R}^n) = H^b(\mathbb{R}^n)$). This characterization is as follows.