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Positive solutions for the Kirchhoff-type problem involving general critical growth – Part I: Existence theorem involving general critical growth

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ABSTRACT

In this paper, we consider the following Kirchhoff-type problem

$$\begin{cases} \left(a + \lambda \int\limits_{\mathbb{R}^3} |\nabla u|^2 dx + \lambda b \int\limits_{\mathbb{R}^3} |u|^2 dx\right)(-\Delta u + bu) = f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \ u > 0, \text{ in } \mathbb{R}^3, \end{cases}$$

where $\lambda \geq 0$ is a parameter, a, b are positive constants and f reaches the critical growth. Without the Ambrosetti–Rabinowitz condition, we prove the existence of positive solutions for the Kirchhoff-type problem with a general critical nonlinearity. We also study the asymptotics of solutions as $\lambda \to 0$. Numerical solutions for related problems will be discussed in the second part.

1. Introduction

This is the first paper in a series. It deals with theory while the second part is concerned with the numerical aspects. In this paper Part I, we consider the existence of positive solutions for the following nonlinear Kirchhoff-type problem

$$\begin{cases} \left(a + \lambda \int\limits_{\mathbb{R}^3} |\nabla u|^2 dx + \lambda b \int\limits_{\mathbb{R}^3} |u|^2 dx\right)(-\Delta u + bu) = f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \ u > 0, \text{ in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $\lambda \geq 0$, a, b are positive constants and the general nonlinearity f has a critical growth. Problem (1.1) arises from an interesting physical background. In fact, if we replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^3$ and let $\lambda = 1$ and $b = 0$, we obtain the following Kirchhoff-type problem

$$\begin{cases} -\left(a + \int\limits_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

which is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int\limits_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.3}$$

Equation (1.3) was first proposed by Kirchhoff in [18] describing the classical D'Alembert's wave equations for transversal oscillations of elastic strings, particularly, taking into account the change in string length caused by vibration. Lions introduced in [22] a functional analysis approach and described the abstract framework to the following problem

$$\begin{cases} u_{tt} - \left(a + b\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases} \tag{1.4}$$

Besides this, problem (1.4) also models several biological systems [1] from a mathematical biological point of view, where u shows a process that depends on the average of itself (for example, population density). For more detailed physical and biological background of Kirchhoff-type problem, we refer the reader to the papers [4,26] and the references therein.

Recently, problem (1.2) has been studied in literatures by variational methods, cf., for example [7,13,14, 27–29,37,39]. These works show an increasing interest in studying the existence of least energy solutions, positive solutions, multiple solutions, sign-changing solutions and semiclassical states. Meanwhile, various solvability conditions on the general nonlinearity f near infinity and zero, for example, the asymptotic case [31] and super-linear case [27], have been considered. Particularly, in [1], Alves, Corrêa and Ma considered problem (1.2) and proved the existence of positive solutions by the Mountain Pass Theorem. In [28], using the Young index and critical groups, Perera and Zhang obtained nontrivial solutions for problem (1.2). With the aid of mini–max methods and invariant sets of decent flow, Zhang and Perera [39], Mao and Zhang [27] studied the existence of three solutions (a sign-changing solution, a positive solution and a negative solution). In [13], He and Zou proved the existence of infinitely many solutions by Fountain Theorems. For more results of (1.2), we refer the reader to [9,10,24].

In terms of the Kirchhoff-type problem in $\mathbb{R}^N$, there are also several existence results, see for example [2,12,15,16,20,21,19,23,25,32,33,35,36] and the references therein. In these works, the existence of positive solutions, mountain pass solutions and high energy solutions were obtained with f satisfying various conditions. In particular, we mention the following two existence results for (1.1) with $\mathbb{R}^3$ replaced by $\mathbb{R}^N$. In [19], Li et al. considered problem (1.1) under the following assumptions:

(f_1) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $|f(t)| \leq c\left(|t| + |t|^{p-1}\right)$ for all $t \in \mathbb{R}_+ = [0, \infty)$ and some $p \in (2, 2^*)$, where $2^* = 2N/(N-2)$, for $N \geq 3$;