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Ground states of nonlinear Schrödinger equation on star metric graphs ☆

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A B S T R A C T

The existence and nonexistence of the ground state to nonlinear Schrödinger equation on several types of metric graphs are considered. In particular, for some star graphs with only one central vertex, the existence of ground state solution or positive solutions are shown. It is shown that the structure of the set of positive solution is quite different from the one for corresponding bounded n-dimensional domain. The proofs are based on variational methods, rearrangement arguments, energy estimates and phase plane analysis.

1. Introduction

The nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + r\Delta\psi + \chi|\psi|^2\psi = 0, \qquad t > 0,\ x \in \mathbb{R}^n, \tag{1.1}$$

arises as a canonical model of physics from the studies of continuum mechanics, condensed matter, nonlinear optics, plasma physics [15,34]. A standing wave solution of (1.1) is in a form of $\psi(x,t) = \exp(\lambda it)\Psi(x)$ and Ψ satisfies a nonlinear elliptic equation:

$$r\Delta\Psi - \lambda\Psi + \chi|\Psi|^2\Psi = 0, \qquad x \in \mathbb{R}^n, \tag{1.2}$$

which has been extensively considered in the last a few decades [9,10,33]. Here r is interpreted as the normalized Plank constant, χ describes the strength of the attractive interactions and λ is the wavelength.

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Standing wave solutions of more general Schrödinger type equations have also been studied in [7,8,14,16,17,20,21,25,26,28,29,36,38].

While the standard spatial setting for the nonlinear Schrödinger equation is the Euclidean space $\mathbb{R}^n$ for $n = 1, 2, 3$, there has been recent interests on wave propagations on thin graph-like domains which can be approximated by metric graphs (or quantum graphs) [11,18,23,32]. A metric graph is a graph $G = (V, E)$ with a set V of vertices and a set E of edges, such that each edge $e \in E$ is associated with either a closed bounded interval $I_e = [0, l_e]$ of length $l_e > 0$, or a closed half-line $I_e = [0, \infty)$ with $l_e = \infty$ in this case. The notion of graph is central to this paper, and we refer the reader to [12,39] for the basic definitions in graph theory. For each edge $e \in E$ joining two vertices $v_1, v_2 \in V$, a coordinate system x_e is chosen along $I_e = [0, l_e]$, in such a way that v_1 corresponds to $x_e = 0$ and v_2 to $x_e = l_e$, or vice versa. In the case that $l_e = \infty$, we always assume that the half-line I_e is attached to the remaining part of the graph at $x_e = 0$, and the vertex corresponding to $x_e = +\infty$ is called a vertex at infinity. The subset of V consisting of all vertices at infinity is denoted by V_∞ [5].

In this paper, we investigate the existence and nonexistence of ground state solutions to a nonlinear Schrödinger (NLS) equation on a connected metric graph $G = (V, E)$:

$$\begin{cases} -u_e'' + u_e = |u_e|^{p-2} u_e, & \text{for each edge } e \in E, \\ \displaystyle\sum_{e \succ v} \frac{du_e}{dx_e}(v) = 0, & \text{for each vertex } v \in V \backslash V_\infty, \\ u_{e_i}(v) = u_{e_j}(v), & \text{if } e_i \succ v \text{ and } e_j \succ v \text{ for some } \ v \in V \backslash V_\infty, \\ u = (u_e) \in H^1(G), & \end{cases} \tag{1.3}$$

where $p > 2$ and $e \succ v$ means that the edge e is incident to a vertex v. In (1.3), the sum of flux from all edges incident at the vertex v is zero, which is the Kirchhoff's circuit law; and second boundary condition that $u_{e_i}(v) = u_{e_j}(v)$ is known as the continuity condition at the vertex v. If the vertex v is an endpoint (only one edge is incident to v), then the Kirchhoff's condition becomes the Neumann boundary condition at v. If $v \in V_\infty$, there is no given boundary condition but we consider the problem in H^1 space hence we must have $\lim\limits_{x_e \to \infty} u_e(x_e) = 0$ for $u_e \in H^1(I_e)$ and $I_e = [0, +\infty)$. Here $L^p(G)$ is the space defined as the set of functions $u : G \to \mathbb{R}$ such that

$$\int_G |u|^p dx := \sum_{e \in E} \int_{I_e} |u_e|^p dx_e < \infty,$$

and $H^1(G)$ is the Sobolev space defined as the set of functions $u : G \to \mathbb{R}$ such that $u = (u_e)$ is continuous on G and $u_e \in H^1(I_e)$ for every edge $e \in E$ with the natural norm

$$\|u\|^2_{H^1(G)} = \int_G (|u'(x)|^2 + |u(x)|^2) dx = \sum_{e \in E} \int_{I_e} (|u_e'(x_e)|^2 + |u_e(x_e)|^2) dx_e.$$

The energy function corresponding to (1.3) is defined by

$$J(u, G) = \frac{1}{2} \sum_e \int_{I_e} (|u_e'(x_e)|^2 + |u_e(x_e)|^2) dx_e - \frac{1}{p} \sum_e \int_{I_e} |u_e(x_e)|^p dx_e, \quad u \in H^1(G). \tag{1.4}$$

A critical point $u \in H^1(G)$ of $J(\cdot, G)$ satisfies that for any $w = (w_e) \in H^1(G)$, we have

$$(J'(u, G), w) = \sum_e \int_{I_e} (u_e' w_e' + u_e w_e - |u_e|^{p-2} u_e w_e) \mathrm{d}x_e = 0.$$

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