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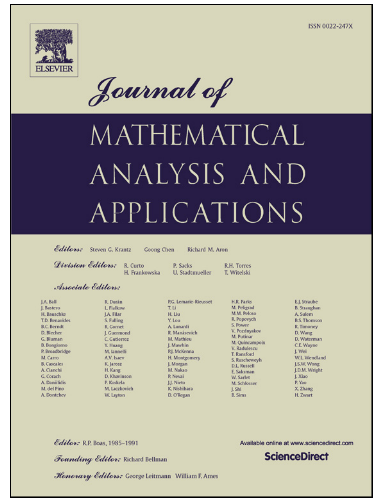

Liouville theorems for the Stationary Navier Stokes equation on a hyperbolic space

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LIOUVILLE THEOREMS FOR THE STATIONARY NAVIER STOKES EQUATION ON A HYPERBOLIC SPACE.

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Abstract. The problem for the stationary Navier-Stokes equation in 3D under finite Dirichlet norm is open. In this paper we answer the analogous question on the 3D hyperbolic space. We also address other dimensions and more general manifolds.

1. Introduction

The familiar Liouville theorem from the complex analysis says that every entire function that is bounded must be constant. Given a PDE, a Liouville-*type* theorem for the PDE, or just a Liouville theorem, is a similar statement: it concludes a solution must be constant if the solution is bounded in some norm. The conclusion of being constant can be broadened to include for example functions that depend on time only if the equation is time dependent.

The Liouville theorems have been studied for many types of PDE: elliptic, parabolic, wave equations, harmonic maps (see for example [11, 20, 16, 21]). Liouville theorems are mathematically interesting to study in their own right, but the strong interest in them stems from the fact that they are an important tool in the regularity theory. Their importance for the Navier-Stokes equations has been illustrated in [23, 17] in connection to ruling out Type 1 singularities.

The Liouville theorem in 2D states

Theorem 1.1. [17] *Let u be a bounded weak solution of the Navier-Stokes equation on $\mathbb{R}^2 \times (-\infty, 0)$. Then $u(x,t) = b(t)$ for some bounded measurable function $b : (\infty, 0) \to \mathbb{R}^2$.*

In 3D, the authors are able to obtain corresponding results for the axi-symmetric equations with no swirl (no swirl means that when written in cylindrical coordinates (r, θ, z), $u_\theta = 0$). They also showed that u is 0 if it is bounded, axisymmetric and in addition $r\,|u|$ is bounded [17]. Other works have been carried out in [19, 10, 2, 3, 12, 13, 15]. Unfortunately, the full 3D problem *without* some additional assumptions on the solutions is still open, even for the stationary Navier-Stokes equation.

Consider the following stationary Navier-Stokes equation on $\mathbb{R}^n$, $n \geq 2$

$$-\Delta u + u \cdot \nabla u + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

together with the conditions that

$$\lim_{|x| \to \infty} u(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla u|^2 < \infty, \tag{1.3}$$

where

$$u : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad p : \mathbb{R}^n \to \mathbb{R}.$$

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