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Modulus of continuity and Lipschitz approximation

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A R T I C L E I N F O

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A B S T R A C T

Given a uniformly continuous map f from a doubling metric space X to a normed linear space V, and given a subadditive function ω, we give a characterization of ω dominating the modulus of continuity of f in terms of Lipschitz approximation. As the main part of this characterization, we give a constructive method of approximating a uniformly continuous map from X to V by Lipschitz maps, the corresponding approximation operation is linear.

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1. Introduction

The modulus of continuity of a map $f : X \to Y$ between two metric spaces is the function $\omega_f : [0, \infty) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$\omega_f(\varepsilon) = \sup\{d(f(x), f(y)) : x, y \in X, d(x, y) \leq \varepsilon\}.$$

The map f is said to be uniformly continuous if there is a $\varepsilon_0 > 0$ such that $\omega_f(\varepsilon) < \infty$ for $\varepsilon < \varepsilon_0$, and $\lim_{\varepsilon \to 0^+} \omega_f(\varepsilon) = 0$. In mathematical analysis, the modulus of continuity of a map is used to measure quantitatively its uniform continuity. In general, it is difficult to determine explicitly the modulus of continuity ω_f of a map f, one is mainly interested in maps whose modulus of continuity can be dominated by a special class of functions. For instance, the inequality $\omega_f(\varepsilon) \leq c\varepsilon$ for some constant c describes the Lipschitz continuity of f, and $\omega_f(\varepsilon) \leq c\varepsilon^\alpha$ $(0 < \alpha < 1)$ describes the Hölder continuity of f. The following property (its a simple proof is given in Section 2) gives a criteria to control ω_f by a prescribed function ω in terms of approximation by Lipschitz maps.

Proposition 1.1. *Let $f : X \to Y$ be a map between two metric spaces and $\omega : [0, \infty) \to [0, \infty)$ an arbitrary function. If for every $0 < \varepsilon < \infty$, there is a Lipschitz map $f_\varepsilon : X \to Y$ such that (i)* $\mathrm{Lip}(f_\varepsilon) \leq c\frac{\omega(\varepsilon)}{\varepsilon}$ *for*

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some constant c; (ii) $\sup_{x\in X} d(f_\varepsilon(x), f(x)) \le c_1\omega(\varepsilon)$ *for some constant* c_1*; then* $\omega_f(\varepsilon) \le c_2\omega(\varepsilon)$ *for some constant* c_2*.*

If $\omega : [0,\infty) \to [0,\infty)$ satisfies $\lim_{\varepsilon\to 0^+} \omega(\varepsilon) = 0$, and $\sup_{x\in X} d(f_\varepsilon(x), f(x)) \le c\omega(\varepsilon)$ for some constant c and $0 < \varepsilon < \infty$, we say that $\{f_\varepsilon\}$ is an ω-approximation of f.

In this paper we shall focus on the problem whether the converse of Proposition 1.1 also holds, namely, the following

Problem 1.2. *Let* $f : X \to Y$ *be a map between two metric spaces. Assume that* ω_f *is dominated by* $\omega : [0,\infty) \to [0,\infty)$ *up to a constant factor and* $\lim_{\varepsilon\to 0^+} \omega(\varepsilon) = 0$*. Under what conditions on* X, Y, ω *and* f*, for every* $0 < \varepsilon < \infty$*, does there exist a Lipschitz map* $f_\varepsilon : X \to Y$ *such that* $\mathrm{Lip}(f_\varepsilon) \le c\frac{\omega(\varepsilon)}{\varepsilon}$ *and* $\sup_{x\in X} d(f_\varepsilon(x), f(x)) \le c_1\omega(\varepsilon)$ *for some constants* c, c_1*?*

In view of Proposition 1.1, an affirmative answer to Problem 1.2 will provide a characterization of ω dominating ω_f.

Removing the quantitative requirement for the Lipschitz constant and approximation error, Problem 1.2 boils down to the usual *Lipschitz approximation problem*:

Problem 1.3. *Given* $\mu > 0$*, under what conditions on* X, Y *and* f*, does there exist a Lipschitz map* $f_\mu : X \to Y$ *such that* $\sup_{x\in X} d(f_\mu(x), f(x)) < \mu$*?*

An affirmative answer to Problem 1.2 is stronger than an affirmative answer to Problem 1.3. Even in the case of Problem 1.3, if there are no restrictions on X, Y or f, we will obtain a negative answer as shown in the examples in [8, p. 6] and [3, p. 18].

Let us recall some positive answers to the usual Lipschitz approximation problem (Problem 1.3). In [8, p. 6], J. Heinonen provides a constructive method to approximate a continuous map from a compact metric space X to $l_\infty(\Gamma)$ by a Lipschitz map, here $l_\infty(\Gamma)$ is the Banach space of all bounded real-valued functions on a set Γ with the usual sup-norm. In [7], it is shown that each continuous map from a compact metric space X to a convex subset of a normed linear space can approximated by a Lipschitz map. Removing the assumption of compactness, in [6] and [2], it is shown that each uniformly continuous function $f : X \to \mathbb{R}$ can be approximated by *Lipschitz in small functions*. Here a function $f : X \to \mathbb{R}$ is said to be Lipschitz in small if there are constants $K < \infty$ and $r > 0$ such that $|f(x) - f(y)| \le Kd(x, y)$ whenever $d(x, y) < r$.

In the monograph of Y. Benyamini and J. Lindenstrauss [3, Chapter 1, 2], we can find several results concerning Problem 1.2 as follows.

Theorem 1.4. [3, p. 35, Prop. 2.1; p. 17, Prop. 1.10] *Let* $f : X \to Y$ *be a uniformly continuous map between two metric spaces, whose modulus of continuity* ω_f *is dominated by a subadditive function* $\omega : [0,\infty) \to [0,\infty)$ *with* $\lim_{\varepsilon\to 0^+} \omega(\varepsilon) = 0$*.*

(1) If $Y = l_\infty(\Gamma)$ *for some set* Γ*, then for every* $0 < \varepsilon < \infty$*, there is a Lipschitz map* $f_\varepsilon : X \to l_\infty(\Gamma)$ *such that* $\mathrm{Lip}(f_\varepsilon) \le \frac{2\omega(\varepsilon)}{\varepsilon}$ *and* $\sup_{x\in X} \|f_\varepsilon(x) - f(x)\| \le 3\omega(\varepsilon)$*.*

(2) If X *is a subset of a Hilbert space and* Y *is a Hilbert space, then the same conclusion as (1) holds.*

(3) If Y *is an absolute Lipschitz retract and* $\omega(\varepsilon) = c\varepsilon^\alpha$*,* $c > 0$*,* $0 < \alpha < 1$ *(i.e.,* f *is* α*-Hölder), then for every* $0 < \varepsilon < \infty$*, there is a Lipschitz map* $f_\varepsilon : X \to Y$ *such that* $\mathrm{Lip}(f_\varepsilon) \le \frac{c_1\omega(\varepsilon)}{\varepsilon}$ *for some constant* c_1 *and* $\sup_{x\in X} d(f_\varepsilon(x), f(x)) \le c^{-1}\omega(\varepsilon)$*.*

In Theorem 1.4 (3), the condition of absolute Lipschitz retract allows us to reduce the target space to the case of $l_\infty(\Gamma)$ by Kuratowski's isometric embedding $Y \hookrightarrow l_\infty(Y)$ (see the remark at the end of this paper or [8, p. 5]) and the corresponding Lipschitz retraction $r : l_\infty(Y) \to Y$ (cf. [3, p. 18]). Furthermore, in Theorem 1.4 (1) and (3), the target space $Y = l_\infty(\Gamma)$ actually boils down to $Y = \mathbb{R}$ by considering

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