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Nonlocal Cahn–Hilliard–Navier–Stokes systems with shear dependent viscosity

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A B S T R A C T

We consider a diffuse interface model for the phase separation of an incompressible and isothermal non-Newtonian binary fluid mixture in three dimensions. The averaged velocity $\boldsymbol{u}$ is governed by a Navier–Stokes system with a shear dependent viscosity controlled by a power $p > 2$. This system is nonlinearly coupled through the Korteweg force with a convective nonlocal Cahn–Hilliard equation for the order parameter φ, that is, the (relative) concentration difference of the two components. The resulting equations are endowed with the no-slip boundary condition for $\boldsymbol{u}$ and the no-flux boundary condition for the chemical potential μ. The latter variable is the functional derivative of a nonlocal and nonconvex Ginzburg–Landau type functional which accounts for the presence of two phases. We first prove the existence of a weak solution in the case $p \geq 11/5$. Then we extend some previous results on time regularity and uniqueness if $p > 11/5$.

1. Introduction

We consider a mixture of incompressible, isothermal and (partially) immiscible binary fluids in a given bounded domain $\Omega \subset \mathbb{R}^3$. We suppose that they both have density equal to one and we denote by $\boldsymbol{u}$ their (volume) averaged velocity and by φ the (relative) concentration difference. A well-known diffuse interface model (see, e.g., [4,25,26]) for the phase separation of the mixture is given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div} \boldsymbol{\mathcal{S}}\big(\varphi, D\boldsymbol{u}\big) + \nabla\pi = \mu\nabla\varphi + \boldsymbol{h}(t) \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \tag{1.2}$$

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$$\partial_t\varphi + (\boldsymbol{u}\cdot\nabla)\varphi = \Delta\mu \tag{1.3}$$

$$\mu = -\Delta\varphi + F'(\varphi) \tag{1.4}$$

in $\Omega\times(0,T)$, $T>0$. Here the mobility and other constants have been taken equal to one, F is a double well potential (e.g., $F(r)=(r^2-1)^2$, $r\in\mathbb{R}$) which accounts for the presence of two components, $\boldsymbol{h}$ is an external force. The stress tensor $\boldsymbol{\mathcal{S}}$, up to the pressure term, depends on the symmetric gradient $D\boldsymbol{u} := (\nabla\boldsymbol{u}+\nabla^T\boldsymbol{u})/2$ of the velocity field $\boldsymbol{u}$ and, possibly, on φ, through a suitable constitutive law. If, for instance, we have

$$\boldsymbol{\mathcal{S}}\big(\varphi, D\boldsymbol{u}\big) = \nu(\varphi)D\boldsymbol{u}\,, \tag{1.5}$$

ν being a given strictly positive function, then we are in presence of a Newtonian mixture. The corresponding system (1.1)–(1.4) is called Cahn–Hilliard–Navier–Stokes system (see, e.g., [1,8,9,20,30,34,36]). When the mixture has non-Newtonian features, then the stress tensor itself depends on some power of $|D\boldsymbol{u}|$. A typical example is given by

$$\boldsymbol{\mathcal{S}}\big(\varphi, D\boldsymbol{u}\big) = \big(\nu_1(\varphi)+\nu_2(\varphi)|D\boldsymbol{u}|^{p-2}\big)\,D\boldsymbol{u}\,, \tag{1.6}$$

where ν_i, $i=1,2$, are strictly positive functions and $p>1$. Concerning the single non-Newtonian fluids see, for instance, [32] for the physical background, and [31] for the basic mathematical theory; cf. also [10,13] and its references for more advanced development. The system (1.1)–(1.4) has also recently been investigated in a number of contributions (see [2,7,23,24,27]). In those papers, the chemical potential μ (see (1.1)) is the functional derivative of the Ginzburg–Landau type functional

$$\mathcal{F}(\varphi) = \int\limits_{\Omega}\left(\frac{|\nabla\varphi(x)|^2}{2} + F(\varphi(x))\right)dx\,.$$

However, this is a phenomenological assumption and a more rigorous approach shows that the functional should be nonlocal (see [21,22]). For instance, following [5], we can take

$$\mathcal{E}(\varphi) = \frac{1}{4}\int\limits_{\Omega\times\Omega} J(x-y)|\varphi(x)-\varphi(y)|^2dxdy + \int\limits_{\Omega} F(\varphi(x))dx\,. \tag{1.7}$$

Here $J:\mathbb{R}\to\mathbb{R}$ is a sufficiently smooth interaction kernel such that $J(x)=J(-x)$. With this choice the chemical potential becomes

$$\mu = a\varphi - J*\varphi + F'(\varphi)\,,$$

where

$$a(x) := \int\limits_{\Omega} J(x-y)dy\,, \qquad (J*\varphi)(x) = \int\limits_{\Omega} J(x-y)\varphi(y)dy\,. \tag{1.8}$$

Therefore we have the following nonlocal system in $\Omega\times(0,T)$

$$\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}\boldsymbol{\mathcal{S}}\big(\varphi, D\boldsymbol{u}\big) + \nabla\pi = \mu\nabla\varphi + \boldsymbol{h}(t) \tag{1.9}$$

$$\operatorname{div}\boldsymbol{u} = 0 \tag{1.10}$$

$$\partial_t\varphi + (\boldsymbol{u}\cdot\nabla)\varphi = \Delta\mu \tag{1.11}$$

$$\mu = a\varphi - J*\varphi + F'(\varphi)\,. \tag{1.12}$$