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An invariant Harnack inequality for a class of subelliptic operators under global doubling and Poincaré assumptions, and applications

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ABSTRACT

The aim of this paper is to prove an invariant, non-homogeneous Harnack inequality for a class of subelliptic operators $\mathcal{L}$ in divergence form, with low-regular coefficients. The main assumption, whose geometric meaning is well known in the literature on Harnack inequalities, is the requirement that $\mathcal{L}$ be naturally associated with a Carnot–Carathéodory doubling metric space, where a Poincaré inequality also holds. Both doubling and Poincaré conditions are assumed to hold globally for every CC-ball: accordingly, the Harnack inequality will hold true on every CC-ball. Applications to inner and boundary Hölder estimates are provided, together with pertinent results on the Green function for $\mathcal{L}$. An explicit example of a class of operators for which our results are fulfilled is also given. Via the Green function for $\mathcal{L}$, the global nature of the Harnack inequality can be applied to the study of the existence of a fundamental solution Γ for $\mathcal{L}$, globally defined out of the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$.

1. Introduction and main results

The aim of this paper is to prove (and to provide applications of) an invariant, non-homogeneous Harnack inequality for a class of subelliptic operators $\mathcal{L}$ (with nonnegative characteristic form, possibly degenerate), under the following divergence form (w.r.t. a density V)

$$\mathcal{L} = \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j}\Big), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

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where $a_{i,j} = a_{j,i}$ are measurable functions (for every $i, j \le N$), $A(x) := (a_{i,j}(x))_{i,j}$ is a positive semidefinite matrix for every $x \in \mathbb{R}^N$, and $V > 0$ is a locally-bounded and measurable function on $\mathbb{R}^N$. Attached with $\mathcal{L}$, we have a natural (Borel) measure, namely

$$\mathrm{d}\mu(x) := V(x)\,\mathrm{d}x, \tag{1.2}$$

where $\mathrm{d}x$ denotes Lebesgue measure on $\mathbb{R}^N$. Throughout the sequel, L^p spaces are meant w.r.t. the measure μ in (1.2) (and we consider only real-valued functions in L^p). The role of V will be further motivated in Remark 1.3.

The main hypotheses on $\mathcal{L}$ that we are about to assume have well-known geometrical meaning in the PDE literature concerning Harnack inequalities: for instance, we assume that $\mathcal{L}$ be naturally associated with a Carnot–Carathéodory (CC, for short) doubling metric space $(\mathbb{R}^N, d, \mu)$ related to a family of vector fields $\{X_1, \dots, X_m\}$, where a Poincaré inequality also holds true.

Both doubling and Poincaré conditions will be assumed to hold *globally* for every CC-ball of $\mathbb{R}^N$; usually in the literature, these conditions are assumed for balls of small radii and with centers confined in a fixed compact set. The positive counterpart of our stronger global assumption, along with the validity of the Harnack inequality globally on every CC-ball of $\mathbb{R}^N$, is motivated by the applications we are aiming at: namely, the study of the *Green function* $g_\Omega(x, y)$ related to $\mathcal{L}$ and to a suitable class of bounded open sets Ω (see Section 5), and (in a forthcoming investigation) the existence of a *global* fundamental solution Γ for $\mathcal{L}$, defined out of the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$, obtained via the Green functions g_{Ω_n} of a family of increasing sets Ω_n invading $\mathbb{R}^N$. The main ingredients to obtain the non-trivial existence of such a Γ are the invariant nature of our Harnack inequality, the well-behaved properties of the Green functions established in the present paper, and the Maximum Principle proved by Gutiérrez and Lanconelli in [16].

We now specify our hypotheses. We assume that the possible degeneracy of the matrix $A(x)$ be controlled by well-behaved vector fields, in the following sense: we assume the existence of a set $X = \{X_1, \dots, X_m\}$ of locally Lipschitz-continuous vector fields, and constants $\lambda, \Lambda > 0$ such that

$$\lambda \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 \le \sum_{i,j=1}^{N} a_{i,j}(x)\,\xi_i\xi_j \le \Lambda \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2, \qquad \forall\ x, \xi \in \mathbb{R}^N. \tag{1.3}$$

Let d_X (or d, for simplicity) denote the CC-distance associated with the family X; the ball $B_d(x, r)$ will also be denoted indifferently by $B(x, r)$ or $B_r(x)$. We assume that $\mathbb{R}^N$ is X-connected, and that d satisfies the following axiomatic assumptions:

(D): if μ is the measure (1.2) associated with $\mathcal{L}$, then $(\mathbb{R}^N, d, \mu)$ is a *doubling metric space*, that is, there exists $Q > 0$ such that

$$\mu(B_d(x, 2\,r)) \le 2^Q\, \mu(B_d(x, r)), \qquad \text{for every } x \in \mathbb{R}^N \text{ and every } r > 0; \tag{1.4}$$

(P): the following *global Poincaré inequality* is satisfied: there exists a constant $C_P > 0$ such that, for every $x \in \mathbb{R}^N$, $r > 0$ and every u which is C^1 in a neighborhood of $B_{2r}(x)$, one has

$$\fint_{B_r(x)} \Big|u(y) - u_{B_r(x)}\Big|\,\mathrm{d}\mu(y) \le C_P\, r \fint_{B_{2r}(x)} \Big|Xu(y)\Big|\,\mathrm{d}\mu(y). \tag{1.5}$$