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Pivot duality of universal interpolation and extrapolation spaces

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ABSTRACT

It is a widely used method, for instance in perturbation theory, to associate with a given C_0-semigroup its so-called interpolation and extrapolation spaces. In the model case of the shift semigroup acting on $L^2(\mathbb{R})$, the resulting chain of spaces recovers the classical Sobolev scale. In 2014, the second named author defined the universal interpolation space as the projective limit of the interpolation spaces and the universal extrapolation space as the completion of the inductive limit of the extrapolation spaces, provided that the latter is Hausdorff. In this note we use the notion of the dual with respect to a pivot space in order to show that the aforementioned inductive limit is Hausdorff and already complete if we consider a C_0-semigroup acting on a reflexive Banach space. If the space is Hilbert, then the inductive limit can be represented as the dual of the projective limit whenever a power of the generator of the initial semigroup is a self-adjoint operator. In the case of the classical Sobolev scale we show that the latter duality holds, and that the two universal spaces were already studied by Laurent Schwartz in the 1950s. Our results and examples complement the approach of Haase, who in 2006 gave a different definition of universal extrapolation spaces in the context of functional calculi. Haase avoids the inductive limit topology precisely for the reason that it a priori cannot be guaranteed that the latter is always Hausdorff. We show that this is indeed the case provided that we start with a semigroup defined on a reflexive Banach space.

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1. The classical Sobolev scale

We start by considering the following generic example. Let $(T(t))_{t\geqslant 0}$ denote the left shift semigroup on the Hilbert space $L^2(\mathbb{R})$ generated by the first derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ defined on the domain $D(\frac{\mathrm{d}}{\mathrm{d}x}) = \{f \in$

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$L^2(\mathbb{R})\,;\ \frac{\mathrm{d}}{\mathrm{d}x}f \in L^2(\mathbb{R})\}$. Writing down the abstract interpolation and extrapolation spaces, see Engel, Nagel [5, Chapter II.5], gives the classical scale of Sobolev spaces

$$\cdots \longrightarrow \mathcal{H}^3(\mathbb{R}) \xrightarrow{i_3^2} \mathcal{H}^2(\mathbb{R}) \xrightarrow{i_2^1} \mathcal{H}^1(\mathbb{R}) \xrightarrow{i_1^0} L^2(\mathbb{R}) \xrightarrow{i_0^{-1}} \mathcal{H}^{-1}(\mathbb{R}) \xrightarrow{i_{-1}^{-2}} \mathcal{H}^{-2}(\mathbb{R}) \xrightarrow{i_{-2}^{-3}} \mathcal{H}^{-3}(\mathbb{R}) \to \cdots$$

where the maps are all continuous. Taking the projective limit of this chain of spaces, i.e., endowing the intersection $\cap_{n\in\mathbb{N}}\mathcal{H}^n(\mathbb{R})$ with the coarsest linear topology which makes the inclusions $\cap_{n\in\mathbb{N}}\mathcal{H}^n(\mathbb{R}) \to \mathcal{H}^k(\mathbb{R})$ for all $k \in \mathbb{N}$ continuous, yields the classical function space

$$\mathcal{D}_{L^2}(\mathbb{R}) = \operatorname{proj}_{n\in\mathbb{N}} \mathcal{H}^n(\mathbb{R})$$

studied by Schwartz [10, § 8, p. 199]. Taking the inductive limit, i.e., endowing the union $\cup_{n\in\mathbb{N}}\mathcal{H}^{-n}(\mathbb{R})$ with the finest linear topology which makes the inclusions $\mathcal{H}^k(\mathbb{R}) \to \cup_{n\in\mathbb{N}}\mathcal{H}^{-n}(\mathbb{R})$ for all $k \in \mathbb{N}$ continuous, yields a subspace of the space of distributions which turns out to be isomorphic to the strong dual of $\mathcal{D}_{L^2}(\mathbb{R})$, i.e.,

$$\mathcal{D}'_{L^2}(\mathbb{R}) \cong \operatorname{ind}_{n\in\mathbb{N}} \mathcal{H}^{-n}(\mathbb{R})$$

in a natural way. Also this space was investigated by Schwartz [10, § 8, p. 200]. Indeed, we have the following commutative diagram

$$\begin{array}{ccccccccc}
L^2(\mathbb{R}) & \xrightarrow{i_0^{-1}} & \mathcal{H}^{-1}(\mathbb{R}) & \xrightarrow{i_{-1}^{-2}} & \mathcal{H}^{-2}(\mathbb{R}) & \xrightarrow{i_{-2}^{-3}} & \mathcal{H}^{-3}(\mathbb{R}) & \longrightarrow & \cdots \\
\big\downarrow \mathrm{id}_{L^2(\mathbb{R})} & & \big\downarrow \Phi_1 & & \big\downarrow \Phi_2 & & \big\downarrow \Phi_3 & & \\
L^2(\mathbb{R}) & \xrightarrow[(i_1^0)']{} & \mathcal{H}^1(\mathbb{R})' & \xrightarrow[(i_2^1)']{} & \mathcal{H}^2(\mathbb{R})' & \xrightarrow[(i_3^2)']{} & \mathcal{H}^3(\mathbb{R})' & \longrightarrow & \cdots
\end{array} \tag{1}$$

where the maps Φ_n for $n \in \mathbb{N}$ are isomorphisms. Our first aim is to see that the corresponding inductive limits are isomorphic. We emphasize that this is not trivial just by having "step-wise" isomorphisms. Indeed, we have for instance $\mathcal{H}^{-n}(\mathbb{R}) \cong L^2(\mathbb{R})$ for each $n \in \mathbb{N}$ but $L^2(\mathbb{R}) \ncong \cup_{n\in\mathbb{N}}\mathcal{H}^{-n}(\mathbb{R})$, which shows that we have to be extremely careful when we "identify" isomorphic spaces.

The suitable notion to address our first aim is that of *equivalent inductive sequences*. Each row in the diagram (1) is a so-called inductive sequence, i.e., a sequence $(X_n, i_n^{n+1})_{n\in\mathbb{N}}$ of Banach spaces X_n and linear and continuous maps $i_n^{n+1}\colon X_n \to X_{n+1}$ for $n \in \mathbb{N}$. Two such inductive sequences $(X_n, i_n^{n+1})_{n\in\mathbb{N}}$ and $(Y_n, j_n^{n+1})_{n\in\mathbb{N}}$ are said to be equivalent, if there are increasing sequences $(k(n))_{n\in\mathbb{N}}$ and $(\ell(n))_{n\in\mathbb{N}}$ of natural numbers with $n \leqslant \ell(n) \leqslant k(n) \leqslant \ell(n+1)$ and linear and continuous maps $\alpha_n\colon Y_{\ell(n)} \to X_{k(n)}$, $\beta_n\colon X_{k(n)} \to Y_{\ell(n+1)}$ such that

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{k(n)} & & \xrightarrow{i_{k(n)}^{k(n+1)}} & & X_{k(n+1)} & \longrightarrow & \cdots \\
& \overset{\alpha_n}{\nearrow} & & \overset{\beta_n}{\searrow} & & \overset{\alpha_{n+1}}{\nearrow} & & \overset{\beta_{n+1}}{\searrow} & \\
\cdots \to Y_{\ell(n)} & & \xrightarrow[j_{\ell(n)}^{\ell(n+1)}]{} & & Y_{\ell(n+1)} & & \xrightarrow[j_{\ell(n+1)}^{\ell(n+2)}]{} & & Y_{\ell(n+2)} \to \cdots
\end{array}$$

commutes. As a matter of fact, two equivalent inductive sequences have isomorphic inductive limits.

We now see that the two inductive sequences in (1) are equivalent and we thus get that

$$\operatorname{ind}_{n\in\mathbb{N}} \mathcal{H}^{-n}(\mathbb{R}) \cong \operatorname{ind}_{n\in\mathbb{N}} \mathcal{H}^{n}(\mathbb{R})'$$

holds. Now we would like to conclude that the dual of a projective limit (=intersection) is equal to the inductive limit (=union) of the duals of the spaces in the sequence. This is indeed true but requires an open

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