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Inequalities for series in q-shifted factorials and q-gamma functions

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The paper studies logarithmic convexity and concavity of power series with coefficients involving q-gamma functions or q-shifted factorials with respect to a parameter contained in their arguments. The principal motivating examples of such series are basic hypergeometric functions. We consider four types of series. For each type we establish conditions sufficient for the power series coefficients of the generalized Turánian formed by these series to have constant sign. Finally, we furnish seven examples of basic hypergeometric functions satisfying our general theorems. This investigation extends our previous results on power series with coefficient involving the ordinary gamma functions and the shifted factorials.

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1. Introduction

In a series of papers [8–10,12–15] we considered the logarithmic convexity and concavity with respect to the parameter μ for the class of functions defined by the series

$$f(\mu;x)=A_0(\mu)\sum_{n\geq 0} f_n A_1(n+\mu)x^n, \tag{1}$$

where the coefficients $A_0(\cdot)$, $A_1(\cdot)$ are chosen from the following nomenclature

$$A_0,A_1\in\left\{1,\Gamma(\cdot),\frac{1}{\Gamma(\cdot)},\frac{\Gamma(a+\cdot)}{\Gamma(b+\cdot)}\right\} \tag{2}$$

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and f_n is a (usually non-negative) real sequence. Here Γ stands for Euler's gamma function and a, b are non-negative parameters. The main motivating examples of functions of the form (1) are the (generalized) hypergeometric functions. Moreover, their derivatives with respect to parameters other than μ are also instances of (1). Further examples of (1) can be given with f_n not expressible in terms of gamma functions. Our papers [8–10,12–15] cover nearly all possible combinations of A_0 and A_1 from the collection (2). Most our results are shaped as follows. Logarithmic concavity (convexity) of $\mu \to f(\mu; x)$ on an interval I is equivalent to non-negativity (non-positivity) of the generalized Turánian

$$\Delta_f(\alpha, \beta; x) = f(\mu + \alpha; x)f(\mu + \beta; x) - f(\mu; x)f(\mu + \alpha + \beta; x) = \sum_{m=0}^{\infty} \delta_m x^m \tag{3}$$

for arbitrary $\alpha, \beta \geq 0$ such that $\mu, \mu + \alpha, \mu + \beta, \mu + \alpha + \beta \in I$. In many cases, however, we were only able to prove non-negativity of $\Delta_f(\alpha, \beta; x)$ for $\mu, \beta \geq 0$ and $\alpha \in \mathbb{N}$, so that our results in such cases are incomplete in the sense that they can probably be still extended to all $\alpha \geq 0$. On the other hand, in most cases we actually demonstrate that, under certain restrictions, all coefficients δ_m have the same sign. This type of results can be termed "coefficient-wise logarithmic concavity/convexity" and can be viewed as a strengthening of usual log-concavity/log-convexity.

The purpose of this paper is to extend our previous results by substituting the nomenclature (2) with $\{1, \Gamma_q(\cdot), [\Gamma_q(\cdot)]^{-1}\}$, where $\Gamma_q(\cdot)$ denotes the q-gamma function, defined in (6) below. To this end, we prove four theorems corresponding to four nontrivial combinations of A_0, A_1 chosen from the above set. We also present a number of corollaries giving two-sided bounds and integral representations for the generalized Turánians (3) and certain product ratios of functions (1). Finally, we furnish seven examples of q-hypergeometric functions satisfying our general theorems. Some results dealing with the Turán type inequalities for q-hypergeometric functions have been recently obtained by Baricz, Raghavendar and Swaminathan in [2,3] and Mehrez and Sitnik in [18,19]. In particular, our Theorem 1 can be seen as a far-reaching generalization of [2, Theorem 3.2], their connection explored in Example 2. Furthermore, our Theorem 3 generalizes some statements from [2, Theorem 3.1] and [19, Theorem 1] which we explore in Example 3. Continued fractions for and the mapping properties of the ratios of the basic hypergeometric functions have been recently studied in [1,3].

2. Definitions and preliminaries

Let us fix some notation and terminology. We will use the standard symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ to denote natural, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$. A positive function is called logarithmically concave (convex) if its logarithm is concave (convex). Next, a function $f : I \to (0, \infty)$ defined on an interval $I \subset (0, \infty)$ is said to be multiplicatively concave if

$$f(x^\lambda y^{1-\lambda}) \geq f(x)^\lambda f(y)^{1-\lambda}$$

for $\lambda \in [0, 1]$ and all x, y such that $x^\lambda y^{1-\lambda} \in I$. It is multiplicatively convex if the above inequality is reversed. In other words this says that $\log(f)$ is concave function of $\log(x)$, i.e. $\log[f(e^x)]$ is concave. If f is continuous, its multiplicative concavity is equivalent to

$$f(\sqrt{xy}) \geq \sqrt{f(x)f(y)}, \quad x, y \in I, \tag{4}$$

which can be termed Jensen multiplicative concavity, GG-concavity or concavity with respect to geometric means [21, section 2.3]. We will need the following elementary lemma.

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