J. Math. Anal. Appl. 460 (2018) 400–410

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# On the Weber integral equation and solution to the Weber–Titchmarsh problem

S. Yakubovich

*Department of Mathematics, Fac. Sciences of University of Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*

ARTICLE INFO

*Article history:*
Received 7 August 2017
Available online 5 December 2017
Submitted by M.J. Schlosser

*Keywords:*
Weber–Orr integral transforms
Mellin transform
Bessel functions
Associated Legendre functions

ABSTRACT

We derive sufficient conditions for the existence of the Weber formal solution of the corresponding integral equation, related to the familiar Weber–Orr integral transforms. This gives a solution to the old Weber–Titchmarsh problem (posed in [4]). Our method involves properties of the inverse Mellin transform of integrable functions. The Mellin–Parseval equality and some integrals with the associated Legendre functions are used.



## 1. Introduction and preliminary results

In [4] E.C. Titchmarsh formally showed that an arbitrary complex-valued function $g(x),\ x \in \mathbb{R}_+$ can be expanded in terms of the following repeated integral

$$g(x) = \frac{x}{J_\nu^2(ax) + Y_\nu^2(ax)} \int_a^\infty C_\nu(xt, xa) t \int_0^\infty C_\nu(t\xi, a\xi) g(\xi) d\xi dt, \tag{1}$$

where $a > 0$, $\nu \in \mathbb{C}$, $J_\nu(z), Y_\nu(z)$ are Bessel functions of the first and second kind [1], Vol. II and

$$C_\nu(\alpha, \beta) = J_\nu(\alpha) Y_\nu(\beta) - Y_\nu(\alpha) J_\nu(\beta). \tag{2}$$

Expansion (1) is related to the familiar Weber–Orr integral expansions of an arbitrary function $f(x)$ as repeated integrals

$$f(x) = \int_0^\infty \frac{t\, C_\nu(xt, at)}{J_\nu^2(at) + Y_\nu^2(at)} \int_a^\infty C_\nu(\xi t, at) \xi f(\xi) d\xi dt, \tag{3}$$

*E-mail address:* syakubov@fc.up.pt.

https://doi.org/10.1016/j.jmaa.2017.12.003
0022-247X/

$$f(x) = \int_a^\infty C_\nu(xt, xa) t \int_0^\infty \frac{C_\nu(t\xi, a\xi)}{J_\nu^2(a\xi) + Y_\nu^2(a\xi)} \xi f(\xi) d\xi dt, \tag{4}$$

which are widely used, for instance, in heat process problems posed in radial-symmetric regions. As we see, expansions (3), (4) are different from (1). Combining with (3), (4), Titchmarsh formally proved (1) for $a = 1$ (see [4], p. 15). He posed the problem to find sufficient conditions for the validity of expansion (1) in order to solve the following Weber integral equation with respect to $g$

$$f(x) = \int_0^\infty C_\nu(x\xi, a\xi) g(\xi) d\xi, \tag{5}$$

where $f(x),\ x \in \mathbb{R}_+$ is a given function. As far as the author is aware this question is still open. Our method will be based on the use of the Mellin transform [5]. Precisely, the Mellin transform is defined in $L_{\mu,p}(\mathbb{R}_+),\ 1 < p \le 2$ by the integral

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx, \tag{6}$$

being convergent in mean with respect to the norm in $L_q(\mu - i\infty, \mu + i\infty),\ q = p/(p-1)$. Moreover, the Parseval equality holds for $f \in L_{\mu,p}(\mathbb{R}_+),\ g \in L_{1-\mu,q}(\mathbb{R}_+)$

$$\int_0^\infty f(x) g(x) dx = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} f^*(s) g^*(1-s) ds. \tag{7}$$

The inverse Mellin transform is given accordingly

$$f(x) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} f^*(s) x^{-s} ds, \tag{8}$$

where the integral converges in mean with respect to the norm in $L_{\mu,p}(\mathbb{R}_+)$

$$||f||_{\mu,p} = \left( \int_0^\infty |f(x)|^p x^{\mu p - 1} dx \right)^{1/p}. \tag{9}$$

In particular, letting $\mu = 1/p$ we get the usual space $L_1(\mathbb{R}_+)$. A special class of functions related to the Mellin transform (6) and its inversion (8), was introduced in [7]. Indeed, we have

**Definition 1** *([7])*. Denote by $\mathcal{M}^{-1}(L_c)$ the space of functions $f(x),\ x \in \mathbb{R}_+$, representable by inverse Mellin transform (8) of integrable functions $f^*(s) \in L_1(c)$ on the vertical line $c = \{s \in \mathbb{C} : \mathrm{Re}(s) = \mu\}$.

The space $\mathcal{M}^{-1}(L_c)$ with the usual operations of addition and multiplication by scalar is a linear vector space. If the norm in $\mathcal{M}^{-1}(L_c)$ is introduced by the formula

$$||f||_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f^*(\mu + it)| dt, \tag{10}$$

then it becomes a Banach space.