J. Math. Anal. Appl. 460 (2018) 425–457

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Riesz transforms in the absence of a preservation condition, with applications to the Dirichlet Laplacian

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A R T I C L E I N F O

Article history:
Received 20 February 2017
Available online 5 December 2017
Submitted by E. Saksman

Keywords:
Harmonic analysis
Riesz transforms
Dirichlet Laplacian

A B S T R A C T

Let L be a linear operator defined such that $-L$ generates an associated heat semigroup e^{-tL}. Suppose this semigroup does not satisfy a preservation condition. A proof that a generalised Riesz transform $Dg(L)$ satisfies $\mathcal{L}^p$ bounds for some range $2 < p < q$ is provided based on two new estimates. One is a Hardy type inequality for L, the other a bound regarding how the gradient of the heat semigroup acts on a characteristic function. Applications are to cases where L is the Dirichlet Laplacian Δ_Ω on inner uniform subsets of $\mathbb{R}^n$.

Contents

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[1] This paper is based on chapter 3 of the author's PhD thesis completed at Macquarie University, Sydney. The author was supported during his PhD by an Australian Postgraduate Award.

https://doi.org/10.1016/j.jmaa.2017.12.001

1. Introduction

Consider a generalised Riesz transform $Dg(L)$ on a space of homogeneous type Ω. The boundedness of such an operator on certain $\mathcal{L}^p$-spaces has been studied by many authors. Given an $\mathcal{L}^2$ Riesz transform bound, conditions for $\mathcal{L}^p$ boundedness, $1 < p \leq 2$, are well known and reasonably minimal [8,14]. The $p > 2$ case is more complex. For $p > 2$ it has been shown [2,4,9], that sufficient conditions for $\mathcal{L}^p$ bounds are Gaussian heat kernel bounds, Gaffney type estimates on the gradient of the heat semigroup, and a preservation condition. The Gaussian heat kernel bound is a relatively stable condition, and the gradient estimates are known to be necessary. We focus on the preservation condition. Slight alterations to the standard Riesz transform can cause the preservation condition to fail. To say that a preservation condition fails is to say:

$$e^{-tL}1 \neq 1.$$

This article presents sufficient conditions that ensure $\mathcal{L}^p$-norm boundedness, $p > 2$, of a generalised Riesz transform $Dg(L)$, in cases where a preservation condition fails. Some new estimates will take the preservation condition's place. All the conditions for this paper are formally listed in Section 1.1, and the resulting theorems in Section 1.2. These theorems are then proven in Section 2 of this paper, and an application is contained in Section 3. The method chosen for the main proof is a variation of the good-λ inequality method as used in [4]. The application includes general results for Riesz transforms based on Dirichlet Laplacians on inner uniform subsets of $\mathbb{R}^n$, and, in particular, includes very specific ranges of p for the $\mathcal{L}^p$-norm boundedness of such Riesz transforms on the exterior of cones in $\mathbb{R}^n$. The author would like to convey many thanks to Prof. Xuan Duong for helpful advice and discussion, and to the referee for their suggestions and careful reading.

1.1. Background and conditions

In this section the generalised Riesz transforms considered in this paper are formally defined, and the conditions needed for the main theorem are given. Let Ω be a measurable subset of a space of homogeneous type, with measure μ and metric d. Further let $\|\cdot\|_p$ denote the standard $\mathcal{L}^p$-norm on Ω, and $\mathcal{L}^p(\Omega)$ be the space of such functions. We suppose $\mu(\Omega) = \infty$, and outline a generalised Riesz transform $Dg(L)$ on Ω as follows. In this outline, we will need the sector

$$S_\omega = \{\xi \in \mathbb{C} : |\arg(\xi)| \leq \omega\} \cup \{0\},$$

and its interior S_ω^0. We will also need for $\nu > \omega$ the subset of holomorphic functions on S_ν^0 given by

$$F(S_\nu^0) = \{g \in H(S_\nu^0) : \text{exists } s > 0, c > 0, |g(\xi)| \leq c\big(|\xi|^s + |\xi|^{-s}\big)\}.$$

Fix $\omega \in (0, \pi/2)$. By a type-ω operator we mean a closed linear operator, with spectrum contained within S_ω, and for every $\nu > \omega$ and $\zeta \notin S_\nu$ satisfying the following bound

$$\|(L - \zeta I)^{-1}\| \leq c_\nu |\zeta|^{-1}.$$

Let L be a type-ω operator, one-one with dense domain and range in $\mathcal{L}^2(\Omega)$. It is known (see, for example, [14] and references therein) that such an operator generates a functional calculus consisting of operators $f(L)$ for all $f \in F(S_\nu^0)$. Importantly, L generates a bounded holomorphic semigroup e^{-zL} for z in the domain $|\arg(z)| < \pi/2 - \omega$, and in general we assume a bounded $H^\infty(S_\nu^0)$ functional calculus for each $\nu > \omega$. This semigroup is needed to build $g(L)$ from L. Fix $\alpha \in [0, 1)$ and $\nu > 0$ and define the following space