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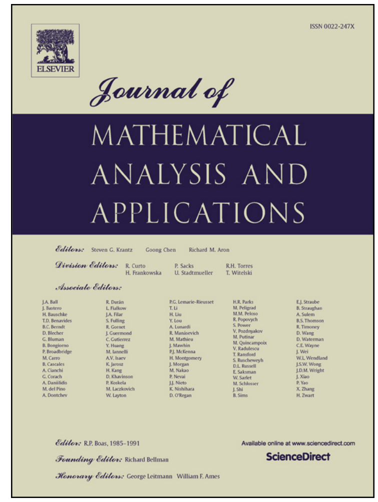

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CHARACTERISATION OF ZERO TRACE FUNCTIONS IN HIGHER-ORDER SPACES OF SOBOLEV TYPE

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Abstract. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a mild regularity property, let $m \in \mathbb{N}$ and $p \in (1, \infty)$, and let $W^{m,p}(\Omega)$ be the usual Sobolev space of order m based on $L^p(\Omega)$; the closure in $W^{m,p}(\Omega)$ of the smooth functions with compact support is denoted by $W_0^{m,p}(\Omega)$. A special case of the results given below is that $u \in W_0^{m,p}(\Omega)$ if and only if all distributional derivatives of u of order m belong to $L^p(\Omega)$ and $u/d^m \in L^1(\Omega)$, where $d(x) = \text{dist}(x, \partial\Omega)$. In fact what is proved is the analogous result when the Sobolev space is based on a member of a class of Banach function spaces that includes both $L^p(\Omega)$ and $L^{p(\cdot)}(\Omega)$, the Lebesgue space with variable exponent $p(\cdot)$ satisfying natural conditions.

1. Introduction

Suppose that Ω is a bounded open subset of $\mathbb{R}^n$, $p \in (1, \infty)$ and d is the distance function defined by $d(x) = \text{dist}(x, \partial\Omega)$ $(x \in \mathbb{R}^n)$; denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the usual first-order Sobolev space $W^{1,p}(\Omega)$. In a recent paper [EN] it was shown that if Ω has a mild regularity property, then $u \in W_0^{1,p}(\Omega)$ if and only if $u \in W^{1,p}(\Omega)$ and $u/d \in L^1(\Omega)$. This was also established when the Sobolev space is based on a Lebesgue space with variable exponent rather than the familiar space $L^p(\Omega)$. Earlier work of this nature had required that u/d should belong to $L^p(\Omega)$ or, more generally, to weak-$L^p(\Omega)$ (see [EE] and [KM]). In the present paper the results of [EN] are extended to higher-order Sobolev spaces based on a certain type of Banach function space.

More details of what is done are desirable. Let $m \in \mathbb{N}$. Given a function $u : \Omega \to \mathbb{R}$ possessing all distributional derivatives of order m, and $j \in \mathbb{N}_0$, write

$$\left|D^j u(x)\right| = \left(\sum\nolimits_{|\alpha|=j} |D^\alpha u(x)|^2\right)^{1/2},$$

where as usual $\alpha = (\alpha_i) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ and $D^\alpha u = \partial^{|\alpha|} u/\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}$. It turns out that if $|D^m u| \in L^1(\Omega)$ and $d^{-m} u \in L^1(\Omega)$, then $d^{k-m} \left|D^k u\right| \in L^1(\Omega)$ for all $k \in \mathbb{N}_0$ with $0 \le k \le m$. Now suppose that Ω has the mild regularity property referred to above, let $X(\mathbb{R}^n)$ be a Banach function space, suppose that $X(\Omega)$ is the corresponding space of functions on Ω obtained from $X(\mathbb{R}^n)$ by the standard process of restriction, and assume that the norm $\|\cdot\|_{X(\Omega)}$ on $X(\Omega)$ has the property that for all $u \in X$,

$$\left\|u\chi_{\Omega(\varepsilon)}\right\|_{X(\Omega)} \to 0 \text{ as } \varepsilon \to 0^+;$$

here χ denotes the characteristic function and $\Omega(\varepsilon) = \{x \in \Omega : d(x) < \varepsilon\}$. This space is isometrically equivalent to a closed subspace of $X(\mathbb{R}^n)$. The Sobolev space of order m based on $X(\Omega)$ is denoted by $W^m(X)$, and $W_0^m(X)$ has the natural meaning. Assume also

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