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Some nonlocal optimal design problems

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ABSTRACT

In this paper we study two optimal design problems associated to fractional Sobolev spaces $W^{s,p}(\Omega)$. Then we find a relationship between these two problems and finally we investigate the convergence when $s \uparrow 1$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, connected and bounded set. For $0 < s < 1$ and $1 < p < \infty$ we consider the fractional Sobolev space $W^{s,p}(\Omega)$ defined as follows

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \colon \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}, \tag{1.1}$$

endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p \, dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{1/p}. \tag{1.2}$$

The term

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$$[u]^p_{W^{s,p}(\Omega)} = [u]^p_{s,p} = \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy, \tag{1.3}$$

is called the *Gagliardo seminorm* of u. We refer the interested reader to [8] for a throughout introduction to these spaces.

The purpose of this paper is to analyze some optimization problems related to the best Poincaré constant in these spaces. First, we consider the following problem: given a measurable set $A \subset \Omega$, we define the optimal Poincaré constant $\lambda_s(A)$ as the number

$$\lambda_s(A) := \inf\left\{ \frac{\frac{1}{2}[v]^p_{s,p}}{\|v\|^p_p} : v \in W^{s,p}(\Omega),\ v = 0 \text{ a.e. in } A \right\}. \tag{1.4}$$

This constant is the largest possible one in Poincaré's inequality

$$\lambda \int_\Omega |v|^p\, dx \le \frac{1}{2} \iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\, dx\, dy$$

for every function $v \in W^{s,p}(\Omega)$ that vanishes on the set A.

Also, this constant can be seen as the first eigenvalue of a fractional p-Laplace type equation. See next section.

The first problem that we want to address is to minimize this constant $\lambda_s(A)$ with respect to the set A in the class of measurable sets of fixed measure. That is, we take $\alpha \in (0,1)$ and define the class

$$\mathcal{A}_\alpha := \{A \subset \Omega \colon A \text{ measurable and } |A| = \alpha|\Omega|\}.$$

So our optimization problem reads, find an *optimal set* $A_s \in \mathcal{A}_\alpha$ such that

$$\lambda_s(A_s) = \Lambda_s(\alpha) := \inf\{\lambda_s(A) \colon A \in \mathcal{A}_\alpha\}. \tag{1.5}$$

This problem is called the *Hard Obstacle Problem* since the optimal set A can be seen as the obstacle where the solution is forced to vanish.

In the case $s = 1$, that is when the classical Sobolev spaces are consider, some related problems were studied in [11,12]. In those papers it was shown that there exists an optimal configuration, and some properties of optimal configurations and of their associated extremals were obtained. We refer to the interested reader to the above mentioned papers.

For this hard obstacle problem, our main result reads:

Theorem 1.1. *Let $\alpha \in (0,1)$, $0 < s < 1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then, there exists a measurable set $A_s \subset \Omega$ such that*

$$|A_s| = \alpha|\Omega| \quad \textit{and} \quad \Lambda_s(\alpha) = \lambda_s(A_s)$$

where $\lambda_s(A)$ and $\Lambda_s(\alpha)$ are given by (1.4) and (1.5) respectively.

Moreover, if $u_s \in W^{s,p}(\Omega)$ is an extremal associated to $\lambda_s(A)$, then

$$A_s = \{u_s = 0\} \cap \Omega \ \textit{a.e.}$$

Related to this optimization problem, is the following variant that sometimes is referred to as the *Soft Obstacle Problem*. That is, given $\sigma > 0$ and $A \subset \Omega$ measurable, we look for the best optimal constant in the