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Existence and multiplicity of positive solutions for Neumann problems involving singularity and critical growth ☆

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ABSTRACT

In this paper, the existence and multiplicity of positive solutions is established for a class of Numann problems of the form

$$\begin{cases} -\Delta u = u^{2^*-1} + \frac{\lambda}{u^\gamma}, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

here Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$, $\gamma \in (0,1)$ and $\lambda > 0$ is a real parameter, $\frac{\partial}{\partial \nu}$ is the outer normal derivative. The main technical approach is based on the variational and perturbation methods.

1. Introduction and main result

The singular bounded value problem of the type

$$\begin{cases} -\Delta u = \mu u^{p-1} + \lambda p(x) u^{-\gamma}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $\gamma \in (0,1), 1 < p < 2^*$ and $\lambda, \mu > 0$ are two numbers, has been investigated by many researchers by imposing different types of hypotheses on λ, μ and p. For example, the existence or uniqueness of positive solutions to problem (1.1) has been studied extensively in the case when $\mu = 0$ (see [6–21] and the references therein). If $\mu \neq 0$, in [18] the authors established two positive solutions to problem (1.1) via the Ekeland's variational principle, provided $1 < p < 2^*$ and $\lambda > 0$ is enough small. In [20], by means of the sub-supersolutions and variational arguments. Yang considered the problem (1.1) with $p(x) = 1$, $p = 2^*$ and $\mu = 1$, obtained that the problem has two positive solutions.

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In addition, in [17–19], the authors proved problem (1.1) possesses at least two positive solutions by the Nehari manifold and careful estimates. In the case when $0 < \gamma \le 1 < p < 2^*$ and $\mu, p(x) = 1$, by using the variational method, Hirano et al. in [8] showed the existence of at least two positive solutions if $\lambda > 0$ is sufficiently small.

Summing up, the similar problem with the Dirichlet boundary conditions has been intensively studied in the literature, however the Neumann problem has attracted less attention. Specifically, Chabrowski in [4] investigated the existence of positive solutions for singular Neumann problem with variational structure:

$$\begin{cases} -\Delta u = u^{2^*-1} + \lambda P(x) u^{-\gamma}, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

here $\lambda > 0$ is a real parameter, $\gamma \in (0, 1)$ is a constant, $P \in C(\overline{\Omega})$ and $P > 0$ on $\overline{\Omega}$. The author proved the existence of at least one solution by means of the approximation and variational methods. After this, Liao et al. in [12] considered the combined effect of critical and singular nonlinearities in Neumann problems of the form:

$$\begin{cases} -\Delta u + u = P(x) u^{2^*-1} + \lambda Q(x) u^{-\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $P, Q \in L^\infty(\Omega)$ are two nonzero nonnegative functions. By using the variational methods, there was a positive solution of the problem with negative energy.

In [5], Chen considered the following singular elliptic equation with Neumann boundary problem:

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2} u}{|x|^s} + \lambda f(x, u), & \text{in } \Omega, \\ D_\gamma u + \alpha(x) u = 0, & \text{on } \partial\Omega \backslash \{0\}, \end{cases}$$

where $N \ge 5$, $0 \le s < 2$, $2^*(s) = \frac{2(N-2)}{N-2}$, $\alpha \in L^\infty(\Omega)$, $\alpha(x) \ge 0$ and f satisfies some suitable conditions, based on the dual fountain theorem, and infinitely many negative energy solutions were established.

Thus, observing the all above studies, an interesting question is whether multiplicity of positive solutions can be established for Neumann problem with negative exponent? Based on the work [2], we shall give some multiplicity results for the following Neumann problem:

$$\begin{cases} -\Delta u = u^{2^*-1} + \lambda u^{-\gamma}, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We define the functional

$$I_0(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx - \frac{\lambda}{1-\gamma} \int_\Omega |u|^{1-\gamma} dx, \quad \forall u \in H^1(\Omega).$$

In general, a function u is called a positive solution of (1.2) if $u \in H^1(\Omega)$ and for all $v \in H^1(\Omega)$ it holds

$$\int_\Omega \nabla u \nabla v dx - \int_\Omega u^{2^*-1} v dx - \lambda \int_\Omega u^{-\gamma} v dx = 0.$$

It is well known that the singular term leads to the non-differentiability of the functional I_0 on $H^1(\Omega)$ (the space $H^1(\Omega)$ is equipped with the norm $\|u\|^2 = \int_\Omega (|\nabla u|^2 + |u|^2) dx$), therefore problem (1.2) cannot