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Note

Complete monotonicity of a function related to the binomial probability

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ABSTRACT

Let k and n be integers with $0 \le k \le n$ and $p \in (0,1)$. We prove that the function

$$G(a) = G_{k,n,p}(a) = \frac{\Gamma(an+1)}{\Gamma(ak+1)\Gamma(a(n-k)+1)} p^{ak}(1-p)^{a(n-k)}$$

is completely monotonic on $(0,\infty)$. This extends a result of Leblanc and Johnson, who showed in 2007 that the sequence $\{G(j)\}_{j=1}^{\infty}$ is decreasing.

1. Introduction

We denote by $f(k,n,p)$ the probability of achieving exactly k successes in n Bernoulli trials with success probability p, that is,

$$f(k,n,p) = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}.$$

As usual, Γ denotes Euler's gamma function. In 2006, Leblanc and Johnson [6] considered the following problem for Bernoulli trials: which is more likely to happen: k successes in n trials or $2k$ successes in $2n$ trials? They proved that for all integers k and n with $0 \le k \le n$ and $p \in (0,1)$, the former is more likely to happen, that is,

$$f(2k,2n,p) \le f(k,n,p). \tag{1.1}$$

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One year later, the same authors [7] presented an interesting generalization of (1.1), as follows. They showed that if $0 \le k \le n$ and $p \in (0,1)$, then

$$f\big((j+1)k,(j+1)n,p\big) \le f(jk,jn,p) \quad (j=1,2,3,\ldots). \tag{1.2}$$

The aim of this note is to provide an extension of this result. We consider the function

$$G(a)=G_{k,n,p}(a)=\frac{\Gamma(an+1)}{\Gamma(ak+1)\Gamma(a(n-k)+1)}p^{ak}(1-p)^{a(n-k)},$$

where a is a positive real number, k and n are integers with $0 \le k \le n$ and $p \in (0,1)$. Inequality (1.2) states that the sequence $\{G(j)\}_{j=1}^{\infty}$ is decreasing. In the next section, we prove that the function G is completely monotonic on $(0,\infty)$.

We recall that a function F is said to be completely monotonic on $(0,\infty)$, if F has derivatives of all orders and satisfies

$$(-1)^N F^{(N)}(x) \ge 0 \quad (N=0,1,2,\ldots;\, x>0).$$

In particular, F is decreasing and convex on $(0,\infty)$. Completely monotonic functions have numerous applications in probability theory, potential theory, numerical analysis and other fields. The main properties of these functions are collected in the monograph [8, chapter IV]. See also the papers [2], [3] and the references cited therein.

2. Results

First, we present two lemmas which play an important role in the proof of our monotonicity theorem.

Lemma 1. *If $c \in (0,1)$ and $y > 1$, then*

$$S_c(y)=\frac{1}{y-1}-\frac{1}{y^{1/c}-1}-\frac{1}{y^{1/(1-c)}-1} > 0. \tag{2.1}$$

Proof. Let $y > 1$. We define $H_c(y)=1/(y^{1/c}-1)$ and set $t=y^{1/c}$. Then,

$$\frac{c^2(t-1)^3}{2t(\log t)^2}\frac{\partial^2}{\partial c^2}H_c(y)=\frac{t+1}{2}-\frac{t-1}{\log t}=\frac{t+1}{2\log t}\int_1^t \frac{(s-1)^2}{s(s+1)^2}ds > 0.$$

It follows that the function $c \mapsto J_c(y)=H_c(y)+H_{1-c}(y)$ is strictly convex on $(0,1)$. Since

$$\lim_{c\to 0+} J_c(y)=\lim_{c\to 1-} J_c(y)=\frac{1}{y-1},$$

we obtain $J_c(y) < 1/(y-1)$ for $c \in (0,1)$. This leads to (2.1). $\square$

The next lemma is a special case of a more general monotonicity theorem given in [4, p. 83].

Lemma 2. *Let $h:(0,\infty)\to(0,\infty)$. If $(-\log h)'$ is completely monotonic on $(0,\infty)$, then h is completely monotonic on $(0,\infty)$.*

We are now in a position to present our main result. The following theorem offers an extension of inequality (1.2).

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